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METHODS OF TREATMENT OF DISPLACEMENT INTEGRAL EQUATIONS

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A summary of application techniques and all of the graphs referred to in this report are bound separately in LA-53-A.

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ABSTRACT

This paper is divided into two parts, LA-53 and LA-53 A. LA-53 treats in great detail the various methods which have been used in the treatment of the physical problems of this project which are representable in terms of integral equations. These problems are primarily those involving the determination of critical sizes and multiplication rates for various configurations of active and tamping materials. A few related problems including age calculations, predetonation probabilities, and a simple albedo problem are also discussed.

A number of graphs have been prepared giving the mathematical data involved in the solutions of these problems and many of the solutions themselves. A brief recapitulation of the methods of solutions of the more standard problems has been prepared. This may be used either separately or in conjunction with the main part of the paper. This recapitulation or "recipe book" and the full collection of graphs compose LA-53 A.

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METHODS OF TREATMENT OF DISPLACEMENT INTEGRAL EQUATIONS

INTRODUCTION

In many problems involving the multiplication and diffusion of neutrons in fissionable and scattering material, integral equations of the type

$$n(\underline{x}) = \int d\underline{x}' \, n(\underline{x}') \, K(|\underline{x} - \underline{x}'|) \, F(\underline{x}')$$

are met. It is proposed to discuss here the properties of equations of this type, the methods of solution which have so far been used, and the results obtained. Equations of this type have been used to describe the physical basis of the determination of the critical sizes and multiplication rates of masses of fissionable material, with or without tampers, and such related problems as the determinations of albedos and detonation probabilities of hypercritical gadgets.

Some of the methods of treatment of the problems discussed here are considerably older than the present problems. The differential diffusion theory was taken over from gas kinetic theory. The simplest from of the extrapolated end-point method was developed in the course of the study of the equation of E. A. Milne describing the flow of radiation through the outer layers of stellar atmospheres. The variation method has been applied to many

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similar problems, and more frequently, as the Ritz method, to differential eigenvalue problems. Many of the mathematical techniques employed here are borrowed from classical probability theory.

In Chapter I the existing body of mathematical theory of displacement integral equations is examined. A large part of this treatment is taken from the review of the subject written by F. Smithies[1]. His treatment is presented in a simplified and less rigorous manner and for the most part transcribed into the notation customary in this project.

In Chapter II the reduction of slab and sphere problems to one-dimensional form is discussed.

In Chapter III these methods are applied to four special cases of displacement integral equations. The first of these is the integral equation with the kernel $K = e^{-|x - x'|}$, which posses a simple exact solution and is therefore a convenient example for displaying the properties common to equations of this type. The second kernel treated is the exponential integral, which is the one-dimensional form of the Milne kernel, which occurs in the most familiar problems of this work. This equation is treated in considerable detail. The remaining two examples are those of the Gauss kernel and the kernel describing the water boiler problem.

In Chapter IV other methods of treatment of these problems and associated problems are discussed. Among these are the variation and numerical

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1) "Singular Integral Equations", London Math. Soc. 46, 409 (1939)

methods of treatment of the integral equations and albedo and non-steady-state problems.

In Chapter V the extension of the end-point method to other shapes than the slab and sphere is considered. The applicability of the method to cylinders and rectangular solids is primarily based on the check afforded by the variation method which has been applied to a few such configurations. The extension of the use of this method to cases where no such check is available is discussed.

In Chapter VI the choice of suitable constants for the integral equation is studied. In the simple form of the integral equation a number of physical simplifications are used. All scattering processes are assumed isotropic and elastic, and the inhomogeneity in energy of the fission spectrum is neglected. The effect of the two approximations is studied to determine appropriate values to use for the cross sections and energy.

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CHAPTER I. THE GENERAL THEORY OF DISPLACEMENT INTEGRAL EQUATIONS

The general type of integral equation which we treat may be written as

$$n(\underline{r}) = c \int d\underline{r}' \, F(\underline{r}') \, n(\underline{r}') \, K(|\underline{r} - \underline{r}'|) \qquad (1.1)$$

where the variable $\underline{r}$ is in one or more dimensions, usually one or three. The integration is to be carried out over all space or over that part of space for which $F(\underline{r}) \neq 0$. In most of the problems treated $F(\underline{r})$ is piecewise constant and of one sign, usually having a value different from zero in only one or two regions. In general there will exist a denumerable (except where $F \neq 0$ over an infinite volume) infinity of eigenvalues, c, one of which is the least. Frequently this least eigenvalue and the corresponding eigenfunction are of primary interest.

The Associated Differential Equation

The simplest problems of this type are those for which $F(\underline{r})$ is constant throughout all space. Although these problems are in themselves of little physical interest their study is of value in that it throws light on the character of the solutions $n(\underline{r})$ in the more interesting problems in regions far removed [i.e. beyond the reach of the kernel $K(|\underline{r} - \underline{r}'|)$] from

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any boundary. For such equations the constant F may be absorbed in the eigenvalue c.

$$n(\underline{r}) = c\int d\underline{r}'\, n(\underline{r}')\, K(|\underline{r} - \underline{r}'|) \qquad (1.2)$$

where the integration is carried over all space. If the kernel K is reasonably regular, the solutions $n(\underline{r})$ of (1.2) are of necessity analytic. In the following it will be assumed that this is the case. Equation (1.2) may be rewritten as

$$n(\underline{r}) = c\int d\underline{r}'\, K(|\underline{r}'|)\, n(\underline{r} + \underline{r}').$$

$n(\underline{r} + \underline{r}')$ may now be expanded as a Taylor series in $\underline{r}'$. Only the even terms of the series will contribute to the integral. For a three-dimensional space the equation then takes the form

$$n(\underline{r}) = c\int d\underline{r}'\, K(|\underline{r}'|)\left[n(\underline{r}) + \frac{1}{2}\left(x'^2\frac{\partial^2 n(\underline{r})}{\partial x^2} + y'^2\frac{\partial^2 n(\underline{r})}{\partial y^2} + z'^2\frac{\partial^2 n(\underline{r})}{\partial z^2}\right) + \ldots\right.$$

$$= c\left(n(\underline{r})\, M_0 + \Delta\, n(\underline{r})\, M_2/3! + \Delta\Delta\, n(\underline{r})\, M_4/5! + \ldots\ldots\right) \qquad (1.3)$$

where M_n is the nth moment of the distribution $K(\underline{r}')$. If $c\, M_0$ is close to one, the second and later terms of the expansion may be small compared with the first. In this case it may be a useful approximation to neglect all terms beyond the second, thus reducing the integral equation to the differential

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diffusion equation

$$\left[(1/3!)(M_2/M_0)\Delta - (1 - cM_0)/(cM_0)\right] n(\underline{r}) = 0 \tag{1.4}$$

As the approximation leading to this diffusion equation is valid only if cM_0 is close to one, equation (1.4) may just as well be written

$$\left[(1/3!)(M_2/M_0)\Delta - (1 - cM_0)\right] n(\underline{r}) = 0 \tag{1.4'}$$

The approximation leading to (1.4) and (1.4') is almost never satisfied in the present work. It is therefore necessary to look for solutions of (1.3).

Since $n(\underline{r})$ is analytic (except, perhaps, at infinity) it can be expressed as a superposition of "wave-functions", $n_k(\underline{r})$, satisfying the equations

$$(\Delta - k^2)\, n_k(\underline{r}) = 0 \tag{1.5}$$

This form of representation of $n(\underline{r})$ is just the Laplace or Fourier transformation which plays a central role in all this theory. It can be seen by substitution that $n_k(\underline{r})$ will satisfy equation (1.3) if and only if

$$c(M_0 + k^2 M_2/3! + k^4 M_4/5! + \ldots\ldots) = 1 \tag{1.6}$$

This is known as the "characteristic equation" of (1.3). The general solution of (1.3) is a superposition of the various wave functions, $n_k(\underline{r})$, for

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all of the values of k satisfying the characteristic equation. For most of the problems which we treat, the characteristic equation has only one solution, i.e., specifies a definite value for $|k|$. The general solution is then a superposition of waves, $e^{\underline{k}\cdot\underline{r}}$, for all vectors, $\underline{k}$, of this magnitude.

The Characteristic Equation

This result is more clearly derived by the use of a Laplace or Fourier transform. Such a transformation is motivated by the fact that the kernel of the integral equation is a displacement operator, a function of $\underline{r} - \underline{r}'$ alone, suggesting an expansion in the eigenfunctions of displacement operators, $e^{\underline{k}\cdot\underline{r}}$. Forming the Fourier transform of equation (1.2), one has

$$
\begin{aligned}
n_k &= \int d\underline{r}\; e^{i\underline{k}\cdot\underline{r}}\, n(\underline{r}) \\
&= c\int d\underline{r}\; e^{i\underline{k}\cdot\underline{r}} \int d\underline{r}'\; n(\underline{r}')\, K(|\underline{r} - \underline{r}'|) \\
&= c\int d(\underline{r} - \underline{r}')\; e^{i\underline{k}\cdot(\underline{r} - \underline{r}')}\, K(|\underline{r} - \underline{r}'|) \int d\underline{r}'\; e^{i\underline{k}\cdot\underline{r}'}\, n(\underline{r}') \\
&= c n_k K_k
\end{aligned}
\qquad (1.7)
$$

where K_k is the Fourier transform of the kernel.

$$
n_k\,(1 - cK_k) = 0 \qquad 1.8)
$$

Thus n_k can differ from zero only where the characteristic equation is satisfied, $1 - cK_k = 0$. This condition, aside from a change of sign in k^2, is identical with (1.6) which is the power series expansion of K_k and may readily be obtained by the expansion of the exponential in its definition.

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It is clear from (1.8) that the above is the only condition imposed by the integral equation; i.e., that the integral equation is satisfied by an arbitrary solution of

$$(\Delta + k^2)\, n_k(\underline{r}) \;=\; 0$$

for any k satisfying the characteristic equation. It is evident from this solution that in the interior of a finite medium the solution of (1.1) has the character of the wave function, $n_k(\underline{r})$, of the symmetry appropriate to the shape of the medium. Near the boundaries the actual solution will deviate from this wave function. The nature of this deviation and the boundary condition thereby imposed on the asymptotic solution $n_k(\underline{r})$ is the subject of the remainder of this chapter.

Solution for Half-Infinite Medium

The simplest case in which to study the boundary effects is that of a "half-infinite" medium, one extending indefinitely on one side of a plane boundary. In this chapter we will treat only the special case in which the solution, $n(\underline{r})$, is a function only of the distance, x, from the boundary. Where there is only one non-zero value of $F(\underline{r})$, the "untamped case, x will be taken positive in that direction. Where F is greater than zero on both sides, the "tamped" case, x will be taken positive on the side on which F is the greater. We will first take F to be zero on one side so that the integration need only be carried over the half-space. In the integral in

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equation (1.2), the two coordinates, y and z, enter only in the kernel and can be integrated out. The integral equation then has the form

$$n(x) = c\int_0^{\infty} dx' \, n(x') \, K(|x - x'|) \tag{1.9}$$

Here the constant c and the kernel K are not necessarily the same as those occurring in the three-dimensional form.

Since in equation (1.9) the integration extends only from zero to plus infinity, it is not immediately clear that the same technique as that of the full open space can be applied. If, however, the function $n(x)$, which is defined by (1.9) for all positive and negative x, is broken up into two parts such that

$$\begin{aligned} n(x) &= f(x) + g(x) \\ f(x) &= 0 \quad \text{for} \quad x > 0 \\ g(x) &= 0 \quad \text{for} \quad x < 0 \end{aligned} \tag{1.10}$$

then the integral equation (1.9) can be written in terms of an integral over the full range of x so that under Laplace transformation it becomes factorable.

$$f(x) + g(x) = c\int_{-\infty}^{\infty} dx' \, g(x') \, K(x - x') \tag{1.11}$$

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Define
$$F(k) \equiv \int_{-\infty}^{\infty} dx\, e^{-kx} f(x)$$
$$G(k) \equiv \int_{-\infty}^{\infty} dx\, e^{-kx} g(x)$$
$$\bar{K}(k) \equiv \int_{-\infty}^{\infty} dx\, e^{-kx} K(x)$$

for those parts of the complex k-plane for which these integrals exist and by analytic extension elsewhere. The Laplace transform of equation (1.11) now becomes

$$\begin{aligned} F(k) + G(k) &= c\int_{-\infty}^{\infty} dx\, e^{-kx}\int_{-\infty}^{\infty} dx'\, g(x')\, K(|x - x'|) \\ &= c\int_{-\infty}^{\infty} d(x - x')\, e^{-k(x-x')}\, K(|x - x'|)\int_{-\infty}^{\infty} dx'\, e^{-kx'}\, g(x') \\ &= c\,\bar{K}(k)\, G(k) \end{aligned} \tag{1.12}$$

This equation has a unique meaning only if there exists a strip parallel to the imaginary axis in which all of the integrals defining these Laplace transforms exist. If this is the case then functions $G(k)$ and $F(k)$ which satisfy (1.12) and are consistent with the restrictions of (1.10) define a unique solution to equation (1.9). It will be assumed in the following that this is the case.

The restrictions imposed on the forms of $n(x)$ and $K(x)$ by this assumption are quite weak. If the value of c is such that the asymptotic solution for $g(x)$ is sinusoidal, i.e. if the characteristic equation has roots only on the imaginary axis, then the integral defining $G(k)$ must exist for all values of k in the right open half plane. This integral extends

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from zero to plus infinity, and the integrand consists of two factors, one of which is bounded, the other decaying exponentially if the real part of k is positive. The character of $f(x)$ for large negative x is determined by the character of the kernel for large values of its argument. For the integral

$$F(k) = \int_{-\infty}^{0} dx\, f(x)\, e^{-kx}$$

to converge in a vertical strip in the right half k-plane it is only necessary that $f(x)$, hence also $K(|x|)$, decay exponentially for large negative x. If then for all $x \geq 0$

$$K(|x|) < N\, e^{-bx}, \quad N > 0, \quad b > 0$$

the integrals defining $F(k)$ and $\bar{K}(k)$ are convergent if the real part of k lies between zero and b, and equation (1.12) has a unique meaning.

If the value of c is such as to give a hyperbolic asymptotic solution, i.e. if the characteristic equation is satisfied for values of k off the imaginary axis, then the asymptotic solution $g(x)$ may increase exponentially for large x. This exponential increase cannot, however, be more rapid than the decay of the kernel or the integral in equation (1.9) will not converge. In this case the integral defining $G(k)$ will not converge throughout the right half k-plane but only for values of k of which the real part is greater than the real part of the root of the characteristic equation determining the asymptotic behavior of $g(x)$. Since, however, the kernel must have

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an asymptotic decay rate greater than the real part of this root the integrals defining $\bar{K}(k)$ and $F(k)$ will converge in a strip in the right half k-plane which overlaps the region of convergence of $G(k)$.

The restrictions imposed to make (1.12) meaningful are therefore satisfied for any kernel which admits a solution of the integral equation. Since in the problems of interest the existence of a solution is guaranteed by the nature of the physical problem, the restriction imposed above will be satisfied in all such problems.

Since the integrands of the integrals defining $F(k)$ and $G(k)$ fail to vanish only for negative and positive values of x respectively, these integrals will correspondingly converge everywhere to the left and right respectively of the common strip of convergence. $G(k)$ will therefore be analytic everywhere to the right of the left hand boundary of this strip, and $F(k)$ everywhere to the left of the right boundary. The analytic extension of $F(k)$, $G(k)$, and $\bar{K}(k)$ may be carried out so as to make equation (1.12) valid for all k. The solution of equation (1.9) is now reduced to the problem of finding two functions $F(k)$ and $G(k)$ satisfying (1.12) and which have a common strip of analyticity and are analytic left and right of this strip respectively. Two such functions are readily found by the following device:

Denote by $P(k)$ the function $c\bar{K}(k) - 1$. Then equation (1.12) reads

$$F(k) = P(k)\, G(k)$$

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hence

$$\ln P(k) = \ln F(k) - \ln G(k) \qquad (1.13)$$

Thus $\ln P(k)$ is expressed as the sum of two parts which are analytic left and right respectively of the common strip except for the roots, if any, of $F(k)$ and $G(k)$. The roots of the characteristic equation are here represented as singularities in $\ln P(k)$. If now a vertical strip containing no singularity of $\ln P(k)$ is chosen, this decomposition can be effected by expressing $\ln P(k)$ as a Cauchy integral.

$$\ln P(k) = (1/2\pi i)\int_C \frac{dk'}{k' - k} \ln P(k') \qquad (1.14)$$

$$= (1/2\pi i)\left[\int_L + \int_R\right] \frac{dk'}{k' - k} \ln P(k')$$

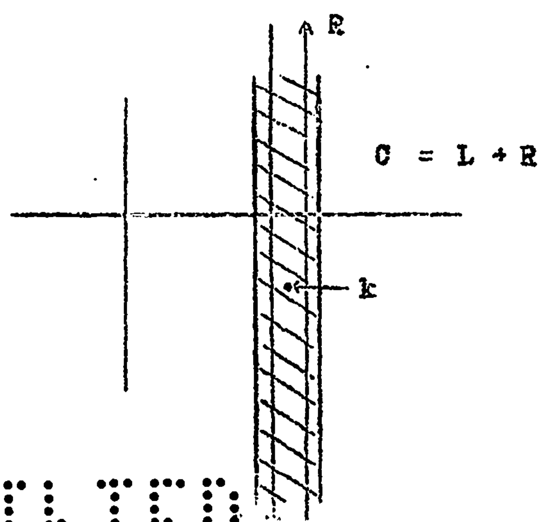

Fig. I

If the point k is within the strip the contours L and R are also within the strip and enclose the point k. If k is outside the strip one or the other contour will be so deformed as to enclose k but no singularity of $\ln P(k')$. The integral over the contour R will therefore be analytic for all k within or to the left of the strip, and the integral over L within or to the right. If now the integral over R is identified with $\ln F(k)$ and the integral over L with $-\ln G(k)$ the conditions required in the decomposition are satisfied. This decomposition is unique once the strip is chosen. This is appropriate since another decomposition also satisfying the conditions imposed must differ from this only in the addition to $\ln G$ and subtraction from $\ln F$ of a function of k which is analytic throughout the entire k plane, i.e. a constant. This change will not affect the character of the solution, $g(x)$, but may be convenient in the evaluation of the integrals. Frequently the constant added and subtracted will be logarithmically infinite. The mathematical transgression required in this process can be avoided if it is so desired by factoring out of $P(k)$ an appropriate polynomial in k so as to make the integrals over L and R separately convergent (cf. F. Smithies[1]).

The solution of (1.9) is then given by

$$\ln G(k) = -(1/2\pi i)\int_L \frac{dk'}{k' - k} \ln\left[c\bar{K}(k') - 1\right] \tag{1.15}$$

$$g(x) \propto \int_{-i\infty+\delta}^{i\infty+\delta} dk\, e^{kx}\, G(k) \tag{1.16}$$

the latter integration being carried up the strip of convergence.

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Extrapolated End-Point

In most of the problems of interest these two integrations can not be performed analytically. In order to find the complete form of $g(x)$ it is therefore necessary to carry out a double numerical integral. As this process is exceedingly laborious, it has not been done. However, a number of important properties of the solution can be found with only a single numerical integral. For a sinusoidal solution $g(x)$ will have the form

$$g(x) = A\left\{\sin\left[k_0(x + x_0)\right] + h(x)\right\} \qquad (1.17)$$

where $h(x)$ approaches zero for large x. Here ik_0 is a root of the characteristic equation. This must be true since far from the boundary the character of the solution is just that of the sine solution of the full-space problem. It is to be expected that the deviation from the asymptotic solution, $h(x)$, will fall off with increasing x about as rapidly as the kernel. The most interesting property of the solution $g(x)$ is the phase of the asymptotic solution, which may be expressed by the extrapolated end-point, x_0.

$$G(k) = \int_0^\infty dx\, e^{-kx}\, g(x)$$

$$= \int_0^\infty dx\, e^{-kx}\, (A/2i)\left[e^{ik_0x_0}\, e^{ik_0x} - e^{-ik_0x_0}\, e^{-ik_0x} + 2ih(x)\right]$$

$$= (A/2i)\left[e^{ik_0x_0}/(k - ik_0) - e^{-ik_0x_0}/(k + ik_0) + 2iH(k)\right]$$

This expansion has poles at $\pm ik_0$. In the neighborhood of these poles $H(k)$ is bounded, hence the logarithm of $G(k)$ is primarily determined by the term which becomes infinite.

$$
\begin{aligned}
\ln G(ik_0 + \epsilon) &= \ln(A/2i) + ik_0x_0 - \ln\epsilon + O(\epsilon) \\
\ln G(-ik_0 + \epsilon) &= \ln(-A/2i) - ik_0x_0 - \ln\epsilon + O(\epsilon)
\end{aligned}
\qquad (1.18)
$$

The difference between these two expressions in the limit of vanishing ϵ is $2ik_0x_0 + \ln(-1)$. In evaluating this limit of the difference from the integral (1.15) giving $\ln G(k)$, we express $\ln G(k)$, the negative L integral, as the R integral minus $\ln P(k)$. The R integral is finite in the neighborhood of the two poles, $\pm ik_0$, as its contour may be taken so as to remain a finite distance from them. $P(k)$ is the Laplace transform of an even function and is therefore itself even. Its derivative is odd, hence $\ln P(ik_0 + \epsilon) - \ln P(-ik_0 + \epsilon)$ is $\ln(-1) + O(\epsilon)$. The two terms $\ln(-1)$ combine to give some multiple of $2\pi i$. The various multiples give extrapolated end-points differing by a half wave-length. It is convenient always to define the extrapolated end-point as the distance beyond the boundary of the first root of the asymptotic solution. Then we have

$$2ik_0x_0 = (1/2\pi i)\int_R dk' \ \ln P(k') \left[1/(k' - ik_0) - 1/(k' + ik_0)\right]$$

$$x_0 = (1/2\pi i)\int_R \frac{dk'}{k'^2 + k_0^2} \ \ln P(k') \qquad (1.19)$$

[redacted]

This integral usually has to be performed numerically.

If the same derivation is followed through for the case of a hyperbolic interior solution, the same result is found for the zero of the asymptotic solution, $\sinh k_0(x + x_0)$.

The technique here described for finding the asymptotic form of the solution, $n(x)$, in the interior of the right hand region may then be summarized as follows: The propagation vector, k_0, of the asymptotic sinusoidal or hyperbolic solution is given by the root of the characteristic equation, $c\,\bar{K}(k) = 1$. The phase of the asymptotic solution is specified by the extrapolated end-point distance, x_0, which is calculated by the use of (1.19).

<u>Value at the Surface</u>

The order of magnitude of the deviation of the true solution from the asymptotic solution in the neighborhood of the boundary can be determined by evaluating $h(0)$, or what is equivalent $-\, n(0)/A$, where A is the asymptotic amplitude. The quantity $n(0)$ can be determined by making use of the fact that the limit as k goes to infinity of $k\,G(k)$ is $n(0)$:

$$\lim_{k\to\infty} k\,G(k) = \lim_{k\to\infty} \int_0^\infty k\,dx\, e^{-kx}\, n(x)$$

$$= \lim_{k\to\infty} \int_0^\infty dy\, e^{-y}\, n(y/k) = n(0) \qquad (1.20)$$

The normalization constant, A, can be gotten by adding the two limiting forms, (1.18). Thus the determination of $h(0)$ can be performed with two numerical integrations.

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In determining $\ln A$ one again requires $\ln G(k)$ as the R integral minus $\ln P(k)$. The sum of the two R integrals is now finite and can be evaluated numerically. $\ln P(k)$, evaluated at $\pm ik_0 + \epsilon$, contributes a finite term and the $-2 \ln \epsilon$. The finite term is just twice the derivative of $P(k)$ at ik_0, hence can be evaluated without an integration.

The limiting value of the logarithm of $k\,G(k)$ for large k can be seen by the transformation

$$k'' = k\kappa$$

$$\ln G(k) = (-1/2\pi i)\int \frac{d\kappa}{\kappa - 1} \ln P(k\kappa)$$

to depend only on the limiting character of P for large argument. This will, in general, be considerably simpler than P itself. Thus the evaluation of $n(0)$ can usually be effected by the evaluation of an analytic integral or a simple numerical integral, or by some analytic device (cf. Chapter III).

By similar methods further details of the character of the deviation, $h(x)$, can be obtained. An example of this technique is given in Chapter III, section 2.

Tamped Half-Infinite Medium

In the preceding sections we have treated the problem of the solution of equation (1.1) with the assumption that $F(x')$ is zero on one side of a plane boundary and has a constant value greater than zero on the other side. The further restriction has been used that the solution, $n(r)$, depends only on the distance from the boundary. We now consider the case in which

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$F(r)$ has different positive values on the two sides of the boundary. We here keep the restriction that the solution is a function of x alone. Integrating out y and z as before, the integral equation takes the form

$$n(x) = c_1 \int_{-\infty}^{0} dx' \, n(x') \, K(|x - x'|) + c_2 \int_{0}^{\infty} dx' \, n(x') \, K(|x - x'|) \qquad (1.21)$$

where for definiteness it will be assumed that $c_2 > c_1$.

We again break up $n(x)$ into a left and right part as in (1.10). (1.21) now becomes

$$f(x) + g(x) = \int_{-\infty}^{\infty} dx' \, K(|x - x'|) \left[c_1 f(x) + c_2 g(x) \right] \qquad (1.22)$$

Again performing a Laplace transformation, we have

$$F(k) + G(k) = \bar{K}(k) \left[c_1 F(k) + c_2 G(k) \right]$$

$$F(k) = G(k) \frac{c_2 \bar{K}(k) - 1}{1 - c_1 \bar{K}(k)} = P(k) \, G(k) \qquad (1.23)$$

The equation is now of exactly the same form as 1.13. The rest of the treatment proceeds in exactly the same way. The solution of the problem is usually somewhat more complicated in this case, owing to the greater complexity of $P(k)$.

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Solution with Transverse Wave

In the above we have found an exact solution to the integral equation (1.1) for a geometry characterized by a single plane boundary. This solution has been obtained subject to the restriction that the "density function", $n(r)$, depends only on the coordinate x. Thus we have found the family of solutions whose asymptotic behavior is that of a plane wave with a propagation vector normal to the boundary. To complete the general solution it remains only to treat the case in which there may be a transverse component of the propagation vector. To do this it is, of course, only necessary to consider one transverse component, say in the y direction. Since the medium is infinite in both directions along the y axis, the y dependence will be factorable. We therefore assume a definite sinusoidal (or hyperbolic) y dependence characterized by a propagation vector, k_y, and then reduce the three-dimensional integral equation to one in one dimension as before. If then

$$n(r) = n(x)\, e^{ik_y y},$$

equation (1.1) becomes

$$n(x) = \iiint dx'dy'dz'\; n(x)\, e^{ik_y(y'-y)}\, K(|\underline{r}-\underline{r}'|)\, F(x'). \qquad (1.24)$$

If $F(x)$ has two positive values as before, we have for the Laplace transform,

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$$F(k) + G(k) = \left[c_1 F(k) + c_2 G(k)\right] K(k') \tag{1.25}$$

where k' is the geometric sum of k and ik_y, i.e. $\sqrt{k^2 - k_y^2}$. The rest of the treatment is now the same as in the simplest case.

To see the character of the change in the extrapolated end-point introduced by the transverse wave, we consider in greater detail the untamped case, $c_1 = 0$. We have then

$$F(k) = P(k)\, G(k)$$

where

$$P(k) = cK(k') - 1$$

Then

$$x_0 = (1/2\pi i) \int \frac{dk}{k^2 + k_0^2} \ln P(k')$$

where $k''^2 = k^2 - k_y^2$, $k_0'^2 = k_0^2 - k_y^2$ and $P(k_0') = 0$. Since $k^2 + k_0^2 = k'^2 + k_0'^2$ and $dk = \frac{k'dk'}{k} = k'dk'/\sqrt{k'^2 + k_y^2} = dk'/\sqrt{1 + k_y^2/k'^2}$

$$x_0 = (1/2\pi i) \int \frac{dk' \ln P(k')}{(k'^2 + k_0'^2)\sqrt{1 + k_y^2/k'^2}} \tag{1.26}$$

Since k_0' has the same numerical value (for the same c) as k_0 in the previous problem the only change in x_0 is that introduced by the square root factor in the denominator. [illegible] problem in which this end-point

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integral has been evaluated (the Milne kernel) the effect of this factor was very slight. The form of the integral in (1.26) indicates that the effect of the transverse wave will usually be such as to diminish slightly the extrapolated end-point. The end-point distance will still be determined primarily by k_0^s, hence by c.

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CHAPTER II. THE SLAB AND SPHERE

In the first chapter an exact solution to the integral equation (1.1) was obtained for all cases involving only a single plane boundary. It has not so far proved possible to find corresponding exact solutions to problems with two parallel plane boundaries as in a tamped or untamped finite slab. It is clear, however, that if the two boundaries are far apart compared with the extent of the kernel a sufficiently accurate solution can be obtained by assuming that the two boundary conditions may be applied indepently. The extent to which this approximation breaks down with decreasing slab thickness can be determined only by comparison of the results so obtained with the results given by methods which for small thicknesses are more accurate. Such comparisons have been obtained and will be discussed in Chapter IV. It suffices to state here that the results of this comparison are such as to indicate that the extrapolated end-point method is a useful tool for almost all of the problems of physical interest.

The use of this method in treating slabs of finite thickness rests on the assumption that the thickness is sufficient to contain three regions; a central region in which the asymptotic behavior of the solution is well established, and two outside regions in which the boundary effects are important. If this is the case, then, for a specified value of c (or of the various values of c) and a specified number of oscillations of the solution,

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the positions of the two boundaries are fixed with respect to the common asymptotic solution in the middle region, hence also the thickness of the slab. If, on the other hand, the thickness of the slab is specified, the equation becomes an eigenvalue problem in c. The untamped slab will have only a discrete spectrum of eigenvalues, c; the infinitely tamped slab will have a continuous as well as a discrete spectrum.

If the weighting function, $F(r)$, has not plane but spherical symmetry, the integral equation for spherically symmetric solutions can be reduced to that of a corresponding plane problem. Taking $F(\underline{r})$ and $n(\underline{r})$ functions only of the radius, r, we have

$$n(r) = \int d\underline{r}' \, F(r') \, n(r') \, K(|\underline{r} - \underline{r}'|)$$

$$= 2\pi \int_{-1}^{1} d\mu \int_{0}^{\infty} dr' \, r'^2 \, F(r') \, n(r') \, K\left(\sqrt{r^2 + r'^2 - 2rr'\mu}\right)$$

$$y^2 = r^2 + r'^2 - 2rr'\mu$$

$$d\mu = -\frac{y\,dy}{rr'}$$

$$rn(r) = 2\pi \int_{0}^{\infty} dr' \, F(r') \, r'n(r') \int_{|r-r'|}^{r+r'} y\,dy \, K(y)$$

$$= \int_{0}^{\infty} dr' \, F(r') \, r'n(r') \left[K_1(|r-r'|) - K_1(r+r')\right] \qquad (2.1)$$

where $\quad K_1(d) = 2\pi \int_{d}^{\infty} y\,dy \, K(y)$

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In the reduction of a plane problem the one-dimensional kernel is

$$\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \; K\left(\sqrt{x^2 + y^2 + z^2}\right) = 2\pi \int_0^{\infty} \rho d\rho \; K\left(\sqrt{x^2 + \rho^2}\right)$$

$$= 2\pi \int_{|x|}^{\infty} y dy \; K(y) = K_1(y)$$

If now we define

$$n(-r) \equiv n(r), \quad F(-r) \equiv F(r), \quad u(r) \equiv rn(r)$$

equation (1.27) can be written as

$$u(r) = \int_{-\infty}^{\infty} dr' \, u(r') \, F(r') \, K_1(|r - r'|) \tag{2.2}$$

which is just the one-dimensional form of the integral equation for a slab having the same F-distribution as that along a diameter of the sphere. It is clear from its definition that $u(r)$ must be odd in r. Thus the solutions of the spherical problem are in one-to-one correspondence with the odd solutions in the corresponding slab problem. For this reason the independent-boundary-condition approximation gives much more accurate results for the fundamental mode of a spherical problem than for the fundamental mode of a slab of the same c value. In the case that has been most extensively studied the error in the radius of the untamped sphere is completely undetectable for all reasonable values of c. It is to be expected that the same will be true for all of the various kernels which are of interest in this work.

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CHAPTER III. EXAMPLES

§1. THE YUKAWA KERNEL

The Yukawa kernel,

$$K(|\underline{r} - \underline{r}'|) \equiv e^{-|\underline{r} - \underline{r}'|}/4\pi|\underline{r} - \underline{r}'|$$

is the Green's function of a simple differential operator, $1 - \Delta$, and therefore the integral equation with this kernel has a simple solution.

Equivalence to Thermal Diffusion Equation

The differential equation to which the the Yukawa kernel is the Green's function describes the diffusion of neutrons after thermalization in a homogeneous hydrogenous material. If neutrons are thermalized at a rate $g(\underline{r})$, have a diffusion constant D (so that the flux of thermal neutrons is $-D \operatorname{grad} n(\underline{r})$), and are absorbed at a rate $a\, n(\underline{r})$; then the steady-state distribution of thermal neutrons, $n(\underline{r})$, is determined by the differential equation

$$D \Delta n(\underline{r}) - a\, n(\underline{r}) = -g(\underline{r})$$

The solution of this equation is

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$$n(\underline{r}) = (1/4\pi D) \int d\underline{r}' \, g(\underline{r}') \, \frac{e^{-|\underline{r} - \underline{r}'|\sqrt{a/D}}}{|\underline{r} - \underline{r}'|}$$

Because of this simplicity of the solutions and the ease of evaluating the integrals involved, this kernel is useful in illustrating the general features of the present theory. If $n(\underline{r})$ is a solution of the integral equation

$$n(\underline{r}) = \int d\underline{r}' \, F(\underline{r}') \, n(\underline{r}') \, e^{-|\underline{r} - \underline{r}'|}/4\pi|\underline{r} - \underline{r}'| \tag{3.1}$$

then it is also a solution of the differential equation

$$\Delta \, n(\underline{r}) = \left[1 - F(\underline{r})\right] n(\underline{r}) \tag{3.2}$$

It is not, however, true that any solution of (3.2) satisfies (3.1). If, for example, $F(r)$ differs from zero only in a definite region then the integral equation requires that $n(r)$ fall off exponentially away from this region. Thus the integral equation requires that its solution satisfy the differential equation and a boundary condition.

If $F(\underline{r})$ and $n(\underline{r})$ depend only on x, equation (3.1) reduces to

$$n(x) = \int dx' \, n(x') \, K_1(|x - x'|) \, F(x') \tag{3.3}$$

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$$K_1(|x|) = 2\pi \int_0^\infty y\,dy\, e^{-\sqrt{x^2+y^2}} \Big/ 4\pi\sqrt{x^2+y^2}$$

$$= \frac{1}{2}\int_{|x|}^{\infty} \frac{\ell\, d\ell\, e^{-\ell}}{\ell} = \frac{1}{2}\, e^{-|x|}$$

$$n(x) = \frac{1}{2}\int dx'\, n(x')\, F(x')\, e^{-|x-x'|} \qquad (3.4)$$

The characteristic equation of this kernel is

$$(c/2)\int_{-\infty}^{\infty} dx\, e^{-kx}\, e^{-|x|}\; c/(1-k^2) = 1$$

We may now solve by the methods of Chapter I the integral equation (3.3) for the case

$$F(x) = 0 \quad \text{for} \quad x < 0$$
$$= c \quad \text{for} \quad x \geq 0, \quad c > 1 + C^2$$

The Laplace transform of the kernel is $c/(1-k^2)$, hence $\ln P(k)$ is $\ln\left[c/(1-k^2) - 1\right] = \ln\left[(C^2+k^2)/(1-k^2)\right]$ which has branch points at ± 1 and $\pm iC$. The appropriate strip of regularity will lie between $+1$ and the imaginary axis. Then

$$\ln G(k) = -(1/2\pi i)\int_L \frac{dk'}{k'-k}\,\ln\left(\frac{C^2+k'^2}{1-k'^2}\right)$$

$$= -\ln\frac{C^2+k^2}{1-k^2} + \frac{1}{2\pi i}\int_R \frac{dk'}{k'-k}\,\ln\left(\frac{C^2+k'^2}{1-k'^2}\right)$$

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The contour of the R integral may now be deformed so as to lie along the real axis, encircling the branch point at $+1$. The real part of the logarithm then makes no contribution to the integral. The difference in phase of the argument of the logarithm on the upper and lower path is $+2\pi i$. The integral is then $\int_1^\infty dk'/(k' - k)$. This is $-\ln(1 - k)$ plus an infinite constant which we discard. (This removal of the infinite constant can be done more rigorously. Cf. Chapter I.)

$$\ln G(k) = -\ln(C^2 + k^2) + \ln(1 - k^2) - \ln(1 - k)$$

$$G(k) = (1 + k)/(C^2 + k^2) \tag{3.5}$$

$$g(x) = (1/2\pi i)\int_{-i\infty + \delta}^{i\infty + \delta} dk\, e^{kx}\,(1 + k)/(C^2 + k^2)$$

For negative x, e^{kx} vanishes at $+\infty$, thus the contour can be extended to the right and as no singularities are enclosed the integral vanishes. For positive x the contour can be extended to the left and the value of the integral is given by the two residues at $\pm iC$.

$$\begin{aligned} g(x) &= (1/2iC)\left[e^{iCx}\,(1 + iC) - e^{-iCx}\,(1 - iC)\right] \\ &= (1/C)\sin Cx + \cos Cx = \sqrt{1 + C^2}/C\,\sin C(x + C^{-1}\tan^{-1} C) \end{aligned} \tag{3.6}$$

This solution exhibits two interesting properties. $g(0) = g'(0) = 1$, thus the boundary condition on the differential equation is very simple, just the

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requirement that the logarithmic derivative of $g(x)$ be 1 at the surface. Also the sinusoidal behavior holds right up to the surface, i.e. $h(x)$ is everywhere zero. Both of these properties can be seen directly from the integral equation. As pointed out above the solution to the integral equation must satisfy the differential equation (3.2) throughout. Thus the sinusoidal behavior must continue to the surface. Since the integral equation (3.4) requires that $n(x)$ decay for negative x just as $e^{-|x|}$, the logarithmic derivative must be one at the surface. For this integral equation the "middle region" in which the asymptotic behavior of the solution is well established extends rigorously to the surface. Thus the independent application of the boundary condition to two surfaces is perfectly accurate. This example serves to give us confidence that in other equations with kernels differing slightly from this one the independent application of the boundary condition may give fairly accurate results. As shown by the solution (3.6) the extrapolated end-point is $C^{-1} \tan^{-1} C$. This can be verified by the use of the formula (1.19).

$$x_0 = (1/2\pi i) \int_R \frac{dk'}{k'^2 + C^2} \ln \frac{C^2 + k'^2}{1 - k'^2}$$

Again the contour may be deformed so as to lie along the real axis and enclose the point $+1$ giving

$$x_0 = \int_1^\infty \frac{dk'}{k'^2 + C^2} = \frac{\tan^{-1} C}{C}$$

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It will be shown in Section 4 of this chapter that this kernel and the associated diffusion equation are of use in the treatment of the water-boiler problem.

§ 2. THE MILNE KERNEL

Derivation of Integral Equation

In the problems of primary interest in this work the kernel is

$$K(|\underline{r} - \underline{r}'|) = e^{-|\underline{r} - \underline{r}'|}/4\pi(\underline{r} - \underline{r}')^2$$

This is the kernel in the integral equation of E. A. Milne describing the flow of radiation in the outer layers of a star. We use this kernel in the integral equations describing the multiplication and diffusion of neutrons in fissionable and scattering material. We treat primarily problems in which the total mean free path of neutrons is the same in all of the materials involved. We here treat the neutrons as monochromatic and the fission and scattering processes as isotropic. We denote by σ the total collision probability per unit path length. σ is the sum of the scattering, fission and absorption probabilities, σ_s, σ_f, and σ_a. We denote by $F(r) = 1 + f(r)$ the quantity $(\nu\sigma_f + \sigma_s)/\sigma$ where ν is the mean number of neutrons emerging from a fission process. $F = 1 + f$ is then the mean number of neutrons emerging per collision from collisions of all types. In the fissionable

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material f is positive. In absorbing tampers f is negative. We speak of a "perfect tamper" as one in which f is zero. The density of neutrons at the point $\underline{r}$ at time t we denote by $n(\underline{r},t)$. These neutrons suffer collisions at a rate σv per unit time. We may consider that $1 + f$ neutrons emerge from each of these collisions and proceed uniformly in all directions. The density of neutrons at $\underline{r}$ and t is determined by the number of neutrons emerging from collisions at all points, $\underline{r}'$, at earlier times $t - |\underline{r} - \underline{r}'|/v$ which arrive at $\underline{r}$ without suffering another collision. The probability of their arrival is given by $e^{-\sigma|\underline{r} - \underline{r}'|}$ multiplied by the inverse square factor, $1/4\pi|\underline{r} - \underline{r}'|^2$. Thus the rate at which neutrons arrive in a unit volume at $\underline{r}$, $vn(\underline{r},t)$, is

$$vn(\underline{r},t) = \int d\underline{r}' \, \sigma v \, F(\underline{r}') \, n(\underline{r}', t - |\underline{r} - \underline{r}'|/v) \, e^{-\sigma|\underline{r} - \underline{r}'|}/4\pi|\underline{r} - \underline{r}'|^2 \qquad (3.7)$$

This equation will have solutions in which the time dependence is exponential,

$$n(\underline{r},t) = n(\underline{r}) \, e^{\gamma t}$$

hence

$$n(\underline{r}', t - |\underline{r} - \underline{r}'|/v) = n(\underline{r}',t) \, e^{-\gamma|\underline{r} - \underline{r}'|/v}$$

then $n(\underline{r},t)$ hence also $n(\underline{r})$, satisfies the integral equation

$$n(\underline{r}) = \sigma \int d\underline{r}' \, F(\underline{r}') \, n(\underline{r}') \, e^{-(\sigma + \gamma/v)|\underline{r} - \underline{r}'|}/4\pi|\underline{r} - \underline{r}'|^2 \qquad (3.8)$$

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It is convenient now to measure distances in terms of the mean attenuation distance, $1/(\sigma + \gamma/v)$. With this unit of length the equation takes the form

$$n(\underline{r}) = \left[1/(1 + \gamma/\sigma v)\right] \int d\underline{r}' \; F(\underline{r}') \; n(\underline{r}') \; K(|\underline{r} - \underline{r}'|) \tag{3.9}$$

where K now represents the Milne kernel.

The one-dimensional form of this kernel is

$$\begin{aligned} K_1(|x - x'|) &= (1/4\pi)\, 2\pi \int_0^\infty \rho\, d\rho \; e^{-\sqrt{x^2 + \rho^2}}/(x^2 + \rho^2) \\ &= (1/2) \int_{|x|}^\infty \frac{dz}{z}\, e^{-z} \qquad (3.10) \\ &= (1/2)\, E(|x|) \end{aligned}$$

Since we have frequent occasion to use this function we use this simplified notation for the exponential integral instead of the customary $-E(-|x|)$. For the characteristic equation we have

$$\begin{aligned} c \int_{-\infty}^{\infty} dx\; e^{-kx}\, \frac{1}{2}\, E(|x|) &= c \int_0^\infty dx\; E(x) \cosh kx \\ &= (c/k) \int_0^\infty dx\, \frac{e^{-x}}{x} \sinh kx = \frac{1}{2}\, (c/k) \ln \frac{1+k}{1-k} = 1 \end{aligned}$$

$$c = k/\tanh^{-1} k \tag{3.11}$$

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The characteristic equation has two real roots if $c<1$, two imaginary roots if $c>1$. c is here $(1+f)/(1+\gamma/\sigma v)$. c will be greater than 1 in the core and will be (in a perfect tamper, i.e. $f=0$) less than or greater than 1 in the tamper as the gadget is hyper- or hypo-critical. For c greater than 1 the roots will lie at $\pm ik_0$, where

$$c = k_0/\tan^{-1} k_0 \qquad (3.12)$$

Graphs of the functions occurring in (3.11) and (3.12) and their reciprocals are given in Fig. V.

<u>Evaluation of the Extrapolated End-Point</u>

Applying to this kernel the extrapolated end-point formula (1.19) for an untamped surface, we have

$$x_0 = (1/2\pi i)\int_R \frac{dk'}{k'^2 + k_0^2}\,\ln\left(\frac{c}{2k'}\ln\frac{1+k'}{1-k'} - 1\right)$$

deforming the contour as before and performing a few simple transformations gives

$$x_0 = \frac{1}{2\pi i}\int_1^\infty \frac{dk'}{k'^2 + k_0^2}\,\ln\left[\frac{1 - \frac{c}{2k'}\left(\ln\frac{1+k'}{k'-1} + \pi i\right)}{1 - \frac{c}{2k'}\left(\ln\frac{1+k'}{k'-1} - \pi i\right)}\right]$$

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$$= \frac{1}{\pi}\int_1^{\infty} \frac{dk'}{k'^2 + k_0^2} \tan^{-1}\left[\pi \Big/ \left(\ln\frac{k'+1}{k'-1} - \frac{2k'}{c}\right)\right]$$

$$= \frac{1}{\pi}\int_0^{1} \frac{ds}{1 + k_0^2 s^2} \tan^{-1}\left[(\pi/2) \Big/ (\tanh^{-1} s - 1/sc)\right] \qquad (3.13)$$

where as before

$$c = (1 + f)\Big/(1 + \gamma/\sigma v) = k_0/\tan^{-1} k_0$$

The integral for x_0 has been evaluated for a number of values of c. It is found that to a very good approximation cx_0 is constant. Since the accuracy with which x_0 was evaluated is considerably greater than the accuracy of most graphs, it is cx_0 rather than x_0 itself which is presented in Fig. VI. The value of x_0 for $c = 1$ is of special interest for two reasons. Since cx_0 is sensibly constant over a long range (rising by less than one percent at $c = 2$) a useful approximation to x_0 is $x_0(1)/c$. The untamped integral equation for this kernel with $c = 1$, the "equation of E. A. Milne", has been the subject of considerable study in the past. E. Hopf gives for the value of $x_0(1)$, .710. This is in agreement with our determination which gives the value .7104. (This number has been more accurately evaluated by G. Blanch at the request of Placzek and Seidel. The value computed was .71044509).

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Extrapolated End-Point at Tamped Boundary

For a tamped boundary the extrapolated end-point is

$$x_o = (1/2\pi i)\int_R \frac{dk'}{k'^2 + k_o^2}\, \ln\left[\frac{1 - \frac{c}{2k'}\ln\frac{1+k'}{1-k'}}{1 - \frac{c'}{2k'}\ln\frac{1+k'}{1-k'}}\right]$$

where $c = (1 + f)/(1 + \gamma/\sigma v) = k_o/\tan^{-1} k_o$

$$c' = (1 + f_t)/(1 + \gamma/\sigma v), \quad f_t = f \text{ of tamper.}$$

Evaluating this integral in the same way as above gives for the extrapolated end-point, if the interior solution behaves as $\sin k_o(x + x_o)$ and the exterior solution as $e^{k_1 x}$,

$$\begin{aligned} x_o &= \tan^{-1}(k_o/k_1)/k_o + (1/\pi)\int_0^1 \frac{ds}{1 + k_o^2 s^2}\left[\tan^{-1}\frac{\pi/2}{\tanh^{-1}s - \frac{1}{cs}} - \tan^{-1}\frac{\pi/2}{\tanh^{-1}s - \frac{1}{c's}}\right] \\ &= \tan^{-1}(k_o/k_1)/k_o - \Delta x_o \end{aligned} \qquad (3.14)$$

The extrapolated end-point is thus expressed as the difference between two terms, one of which is simple, the other small. The first term is just the value of the extrapolated end-point which would be obtained by assuming the asymptotic solutions to hold up to the boundary on each side and requiring continuity of the logarithmic derivative, i.e. just the diffusion-theoretic

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boundary condition. The second term is not only small but also, over the interesting range of values of k, essentially constant. As may be seen in Figs. VII and VIII a reasonably good approximation is afforded by taking $c_t \Delta x_0$ to be about .045. This quantity is the decrease of the extrapolated end-point as compared with its diffusion-theoretic value measured in units of the mean free path divided by $1 + f_t$. This constancy is good, in the final results of most problems, to about one percent. It is most seriously in error for a very large core (hence small k_0) and a nearly perfect tamper.

Accuracy of the End-Point Method

Since the Milne kernel differs only slightly from the Yukawa kernel it is to be expected that the application of the end-point method to slabs and spheres should give fairly accurate results. The end-point results for the untamped slab have been compared with the results of a parabolic variation calculation performed by H. Bethe and with an asymptotic formula correct in the limit of small thickness. The result of this comparison is presented in Fig. IX. The end-point method is seen to be good to within one percent in the interesting range. A parabolic variation calculation for the untamped sphere was carried out by Pryce and by R. Feynman. The effect of the inclusion of a quartic term was investigated by Pryce showing the quadratic result good to one part in 50,000. No difference between these results and those of the end-point method was detectable except for the limiting value of aF (a = radius) for small a for which the end-point method gives a result incorrect by about one percent. We therefore believe that for untamped and infinitely tamped spheres the end-point method can be used with

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perfect confidence. The same will be true of slabs if an error of one percent is not considered serious. It will be seen in subsequent sections that the end-point method gives very accurate results in problems far beyond the range of problems for which it was designed or for which any obvious theoretical reason for its accuracy exists.

<u>End-Point Method for Finitely Tamped Spheres</u>

In particular the extension of the end-point method to spheres of finite tamper proves to be quite accurate, as verified by a numerical iteration solution. The technique of applying the end-point method to finitely tamped spheres is as follows: The integral equation gives a relationship connecting the radius of the core, the tamper thickness, the multiplication rate, and the values of f in core and tamper. Any four of these five quantities may be specified, the equation then determines the value of the fifth. The most convenient of these to leave unspecified is the radius of the core. As shown in Chapter I the spherical problem is equivalent to the determination of the odd solution in a corresponding slab problem in which a slab of core material of thickness equal to the diameter of the spherical core lies between two slabs of tamper material of thickness equal to the thickness of the spherical shell tamper. If the tamper thickness, multiplication rate, and core and tamper materials are specified then the forms of the "asymptotic" solutions in the core and tamper are fixed as follows: The "solution" in the core is $\sin k_o x$, where x is measured from the center of the slab of core material and k_o is determined from the specified f_{core} and multiplication rate by equation (3.12). The "solution" in the tamper is

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$\sinh k_1(T + a + \Delta x_1 - x)$ where k_1 is determined from the tamper properties by (3.11), T is the tamper thickness, a the core radius, Δx_1 the extrapolated end-point at the outer edge of the tamper as given by Fig. VI. At the core-tamper interface, $x = a$, the phase of the hyperbolic sine is then fixed. Its logarithmic derivative is $-k_1 \coth k_1(T + \Delta x_1)$. If a diffusion-theoretic core radius, a_o, is then determined by equating this logarithmic derivative to k_o ctn $k_o a_o$, the end-point radius is $a = a_o + \Delta x_o$ where Δx_e is given by Figs. VII and VIII. Since the numerical solutions (iterative) which supply the check on the accuracy of this approximation were carried out very early in the present study great accuracy was not required of them. They were good to about one percent in the eigenvalue. The check is therefore less precise than the expected accuracy of this approximation. Even this check, however, is sufficiently precise for all practical purposes.

Value of the Solution at the Surface

In the special case of the nonmultiplying untamped equation, ($f = 0$, with neutrons introduced at infinity) the equation of E. A. Milne, a study has been made of the character of $h(x)$, the difference between the actual and asymptotic solutions. In this case the asymptotic solution is linear in x. The end value, $n(0)$, was determined by the method outlined in Chapter I. If the normalization is such that the asymptotic solution has unit slope, so that $n(x) \rightarrow x + .710$ for large x, then $n(0)$ has the value .5773. This strongly suggests that $n(0)$ is $1/\sqrt{3}$. This is actually the case, as shown by E. Hopf by the following method[2]. The integral equation

2) Monthly Notices of the Roy. Astron. Soc. 90, 287 (1930)

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here is

$$n(x) = \frac{1}{2}\int_0^\infty dx'\, n(x')\, E(|x - x'|) = L\, n(x) \tag{3.15}$$

where E is defined in (3.10) and L represents the integral operator. If $n(x) = x + \phi(x)$ this equation may be written,

$$\phi(x) = L\phi(x) - \frac{1}{2}\int_{-\infty}^{0} dx'\, x'\, E(|x - x'|)$$

$$= L\,\phi(x) + \frac{1}{2}E_3(x)$$

where $\quad E_n(x) = \int_1^\infty dy\, e^{-xy}/y^n$

hence formally

$$\phi(x) = \left(\frac{1}{1 - L}\right)\frac{1}{2}E_3(x) \tag{3.16}$$

$$\cong (1 + L + L^2 + L^3 + \ldots\ldots)\,\frac{1}{2}E_3(x)$$

Differentiating (3.15) gives

$$n'(x) = 1 + \phi'(x) = -\frac{1}{2}\int_0^\infty dx'\, n(x')(d/dx')\, E(|x - x'|)$$

$$= \frac{1}{2}n(0)\, E(x) + \frac{1}{2}\int_0^\infty dx'\, n'(x')\, E(|x - x'|)$$

$$\phi'(x) = \phi(0)\,\frac{1}{2}E(x) + L\,\phi'(x) - \frac{1}{2}\int_{-\infty}^{0} dx'\, E(|x - x'|)$$

$$= L\,\phi'(x) + \frac{1}{2}E(x)\,\phi(0) - \frac{1}{2}E_2(x) \tag{3.17}$$

$$\phi'(x) = \left(\frac{1}{1 - L}\right)\left(\frac{1}{2}E(x)\,\phi(0) - \frac{1}{2}E_2(x)\right)$$

[redacted]

The flux equation from which (3.15) is derivable is

$$\int_x^\infty dx'\; n(x')\; E_2(|x - x'|) - \int_0^x dx'\; n(x')\; E_2(|x - x'|) = \text{const.} \qquad (3.18)$$

The derivative of this equation is (3.15). Evaluating the constant for large x, where $n(x')$ is well represented by x' + constant, gives for the value of the constant in equation (3.18)

$$2\int_0^\infty x\,dx\; E_2(x) = 2/3$$

Then (3.18) for $x = 0$ becomes

$$\int_0^\infty dx\left[x + \phi(x)\right] E_2(x) = 1/3 + \int_0^\infty dx\; \phi(x)\; E_2(x) = 2/3$$

We now can evaluate by two different methods the integral

$$\int_0^\infty dx\; \phi'(x)\; E_3(x) = -\phi(0)\; E_3(0) + \int_0^\infty dx\; \phi(x)\; E_2(x)$$

$$= -\frac{1}{2}\,\phi(0) + 1/3$$

$$\int_0^\infty dx\; \phi'(x)\; E_3(x) = \int_0^\infty dx\; E_3(x)\left(\frac{1}{1 - L}\right)\left(\frac{1}{2}\,E(x)\,\phi(0) - \frac{1}{2}\,E_2(x)\right)$$

$$= \frac{1}{2}\int_0^\infty dx\left(E(x)\,\phi(0) - E_2(x)\right)\left(\frac{1}{1 - L}\right) E_3(x)$$

since the operator L or any power of L is symmetric.

[redacted]

[redacted]

$$\int_0^\infty dx\, \phi'(x)\, E_3(x) = \int_0^\infty dx \left[E_1(x)\,\phi(0) - E_2(x)\right] \phi(x)$$

$$= 2\phi(0)\left[\phi(0) - \frac{1}{2} E_3(0)\right] - 1/3$$

$$= 2\phi^2(0) - \frac{1}{2}\,\phi(0) - 1/3$$

$$2\phi^2(0) = 2/3$$

$$\phi(0) = 1/\sqrt{3}$$

This number, $1/\sqrt{3}$, plays an important role in this theory. It was used for some time for $\phi(0)$ and for the extrapolated end-point as a result of the following argument, owing to Fermi: If $n(x)$ is well represented by a solution of the form

$$n(x) = a + x$$

then the flux at the surface is

$$\frac{1}{2}\int_0^\infty dx\,(a + x)\, E_2(x) = \frac{1}{4}\, a + 1/6$$

If this is equated to the asymptotic flux, 1/3, it gives $a = 2/3$. Using the same expression for $n(x)$ in the integral equation (3.15) to compute a second approximant to $n(0)$, one has

$$n(0)_2 = \frac{1}{2}\int_0^\infty dx\, E(x)(a + x) = \frac{1}{2}\, a + \frac{1}{4}$$

If this is equated to a, the value of $n(0)$ in the first approximation,

[redacted]

then $a = 1/2$. Since in each of these two calculations some feature of the first approximant is identified with the corresponding feature of the second approximant it is not surprising that the two results are somewhat different. It seems plausible to equate the ratio of the two quantities, flux/$n(0)$, in first and second approximation. This gives

$$\frac{(1/4)a + 1/6}{(1/2)a + 1/4} = \frac{1/3}{a}$$

$$a = 1/\sqrt{3}$$

Angular Distribution of Flux at the Boundary

This arugment thus gives correctly the end value of $n(x)$ but not its asymptotic linear form. This is presumably because the argument depends primarily on the local linear approximation to the solution rather than on the character of $n(x)$ for large x. If this is true then it is to be expected that an approximate solution of the form $x + 1/\sqrt{3}$ would give fairly accurately the angular distribution of emerging neutrons (radiation). The distribution in angle-cosine, μ, of the flux at the boundary is

$$f(\mu) = \int_0^\infty dx\, n(x)\, e^{-x/\mu}$$

Taking $n(x) = x + 1/\sqrt{3}$ gives

$$f(\mu) = \int_0^\infty dx\,(x + 1/\sqrt{3})\, e^{-x/\mu} \propto (\mu + \sqrt{3}\,\mu^2) \qquad (3.19)$$

This result, also owing to Fermi, was checked by calculating a few values of $G(k) = f(1/\mu)$. The comparison is shown in Fig. X (cf. also G. Placzek, MT-6). As may be seen from this graph the angular distribution of emergent neutrons or radiation is exceedingly well fitted by the Fermi approximation. The ratio of normal to total flux which is used for calibration is fitted to a few tenths of one percent.

Character of Solution Near the Boundary

The values of $G(k)$ calculated for the purpose of this comparison were used to obtain an estimate of the discrepancy term, $h(x)$. This discrepancy seems to be fairly well fitted for most of its range by an exponential, $.109\, e^{-2.45\,x}$. The approximate accuracy of this fit is indicated by comparing the integral of this approximation to $h(x)$

$$\int_0^\infty dx \;\; .109\, e^{-2.45\,x} \;=\; .0445$$

with the true value of the integral which can be gotten directly from the expansion of $G(k)$ about $k = 0$. The correct value of the integral is .04766 which differs from the above value for the integral by 7 percent. This approximate fit to $n(x)$ is given in Fig. XI.

The rapid decay of $h(x)$ is a very encouraging result in that it indicates that the "middle region" in which the asymptotic behavior of the

 [redacted]

solution is well established will actually exist in most problems of interest. The decay of the discrepancy term is undoubtedly still more rapid for a sinusoidal solution as evidenced by the great accuracy of the end-point solution for spheres of very small radius.

§ 3. THE GAUSS KERNEL

Derivation of Gauss Kernel by Compounding Many Elementary Distributions

The kernel of an integral equation for the diffusion of neutrons represents the distribution function of the displacement suffered by a neutron between successive events. These events need not be collisions of any type but may be more widely spaced events. If the displacement occurring between the two significant events is the vector sum of any displacements small compared with the overall displacement, the distribution function will be approximately Gaussian. This may be seen from the fact that the Fourier transform of the distribution function of the resultant of many displacements is the product of the Fourier transforms of the distribution functions of the individual displacements (these distributions being assumed independent). These individual distribution functions cover a small range in x and therefore fall off slowly with k in Fourier transform. The product of many of these will fall off rapidly with k so that the individual transforms are well represented by two terms of their power series, $1 - a_n k^2 - \ldots$. (For a normalized distribution the constant must be unity.) The product of

 [redacted]

[redacted]

many such binomials is approximately $e^{[illegible]}$ in the range of k for which $a_r k^2$ is small for every r. The exponent of the Gauss function is large compared with any one of these; so the Gauss function is essentially dead beyond this range. This Gauss function is the Fourier transform of the overall distribution function, which is therefore also Gaussian. In this derivation it is assumed that the successive displacements are independent, which can be true only if the distribution of displacements is independent of position. This treatment will therefore be applicable to neutron diffusion problems only if the core and tamper materials are identical as regards the elementary displacements and if the correlation in direction and lengths between successive paths may be neglected. For the water-boiler problem with a water tamper, the first of these conditions will be approximately satisfied. The second condition and the requirement that no one of the elementary displacements be comparable with the overall displacement is not well satisfied. The effect of this change is discussed in the next section. The value of the study of the Gauss kernel lies in the fact that it is a reasonably good approximation in many problems and has no free parameters except for the scale of size.

<u>Extrapolated End-Point Treatment</u>

In the extrapolated end-point treatment of the Gauss kernel a feature is met which does not occur in the preceding examples. The Laplace transform has the form e^{ck^2} which is equated to a constant in the characteristic equation. Thus ck^2 is the logarithm of this constant, which has a denumerable infinity of values. The characteristic equation has an

[redacted]

[redacted]

infinity of roots distributed along a hyperbola in the k-plane. This distribution of roots still permits a vertical strip of regularity; so no difficulty arises in the end-point determination.

The normalized three-dimensional Gauss function is

$$\frac{r_o^3}{(3\pi/2)^{3/2}} \; e^{-3r^2/2r_o^2}$$

where r_o^2 is the mean square displacement of this distribution. If the distribution function of the resultant of many elementary displacements is to be represented by a Gauss function, r_o^2 must be the sum of the elementary mean square displacements.

In the calculations which were carried out with the Gauss function the unit of length was taken to be $(2r_o^2/3)^{1/2}$. In these units the distribution is

$$1/(\pi)^{3/2} \; e^{-r^2}$$

Its one-dimensional form, in which y and z are integrated out, is

$$1/(\pi)^{1/2} \; e^{-x^2}$$

which has for its Laplace transform

$$e^{k^2/4}$$

[redacted]

The characteristic equation is

$$c\, e^{k^2/4} - 1 \equiv P(k) = 0 \qquad (3.20)$$

The roots of this equation occur at

$$k = 2(-\ln c + 2n\pi i)^{1/2} \qquad n = 0, \pm 1, \pm 2, \ldots\ldots$$

If $\quad k = s + it, \quad$ then

$$-2 \ln c = s^2 - t^2$$

so the roots lie on a rectangular hyperbola in the k-plane with axes along the real and imaginary axes. If c is greater than 1 there will be two solutions on the imaginary axis, $\pm ik_0$, hence there will exist a sinusoidal asymptotic solution. If c is less than 1 there will be real roots, $\pm k_1$, which determine the asymptotic solution.

The case of primary interest, for example in an approximate treatment of the water-boiler, is that for which $c > 1$. Here the strip of regularity used in defining $F(k)$ and $G(k)$ is that lying to the right of the roots on the imaginary axis and to the left of the next roots in the right half plane. Where $c = 0$ for $x < 0$, the extrapolated end-point is

$$x_0 = (1/2\pi i) \int_{\Pi} \frac{dk}{k^2 + k_0^2} \ln P(k)$$

[redacted]

$\ln P(k)$ may be written as $\ln(e^{(k^2+k_o^2)/4}-1)$. Since the contour runs to the right of the roots at $\pm ik_o$, we may add to $\ln P(k)$ any term which is analytic in the right half plane without changing the value of the integral. It is convenient to replace $\ln P(k)$ by $\ln\left[P(k)/(1/4)(k^2+k_o^2)\right]$. This makes no change in the value of the integral which now may be evaluated conveniently by numerical integration up the imaginary axis. x_o has been evaluated roughly for a number of values of c. The results are presented in Fig. XII. Several numerical solutions for spherical problems have been obtained (cf. Chapter IV), and give radii in agreement with those of the end-point calculation within half a percent.

Value of the Solution at the Surface

The end value, $n(0)$, has been computed for the linear solution, $c=1$, $n(x) = x + .410 + h(x)$. It has the value 1/2. Thus the character of the deviation, $h(x)$, differs markedly from that for the Milne kernel. In particular, the two are of opposite sign. This change in sign of $h(x)$ is not surprising since the general appearance of the Yukawa kernel (for which $h(x)\equiv 0$) is intermediate between that of the Milne and Gauss kernels.

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§4. THE CHRISTY KERNEL

Derivation of the Yukawa Kernel as the Distribution Function of Overall Displacements Compounded from Elementary Displacements Exponentially Distributed in Number

A reasonably accurate treatment of the water-boiler problem must take into account the fact that the elementary displacements are not negligible in comparison with the overall displacement. In particular, the mean square displacement in the first mean free path is of the order of half of the total mean square displacement in cooling neutrons to thermal energy. Since most of the collisions are with hydrogen the correlation in direction and magnitude of successive elementary displacements will be significant. The diffusion of neutrons after thermalization will not be Gaussian since the number of elementary displacements as well as their magnitudes is statistically distributed. The distribution of displacements in the diffusion after thermalization has been shown to be that of the Yukawa kernel (cf. Section 1). A demonstration of this fact by statistical arguments is as follows:

For a definite number of elementary displacements the distribution function of the overall displacement is $(4\pi a)^{-3/2}\, e^{-r^2/4a}$ with the Laplace transform, e^{ak^2}. Here a is proportional to the number of elementary displacements before capture. This number of elementary displacements will be exponentially distributed since the probability of capture is the same at each step. Averaging the Laplace transform of the Gauss distribution with

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[redacted]

the weighting function e^{-a} gives

$$\int_0^{\infty} da\, e^{-a}\, e^{ak^2} = 1/(1-k^2)$$

which is the Laplace transform of the Yukawa kernel.

In the simplest type of the water-boiler problem we have a spherical core of a water solution of fissionable material surrounded by a pure water tamper. The fissionable material in the core absorbs thermal neutrons and emits fast neutrons. The fissionable material of the core is present in sufficiently low concentration that the absorption of neutrons before thermalization is negligible. The fast neutrons produced by fission are scattered in the water of the core and tamper and slowed to thermal energy. Once thermalized the neutrons diffuse in the water until captured, either by hydrogen or the fissionable material. Since the diffusion distance is different in the core and tamper, owing to the absorption by the fissionable material, it would appear that the distribution of displacements between thermalization and capture could not be described by a displacement kernel. The problem can, however, be formulated as a displacement integral equation by the following device, owing to R. Christy and R. Feynman:

Derivation of the Christy Kernel for the Water-Boiler

Denote by $m(\underline{x})$ the rate at which neutrons are thermalized per unit volume at $\underline{x}$ and by $n(\underline{x})$ the density of thermal neutrons. Take for the unit of time the mean lifetime of thermal neutrons in the core material.

[redacted]

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Then the rate of production of fast neutrons in the core is $n(\underline{r})\,\nu\,c$, where c is the fraction of all thermal neutrons absorbed in the core material which are absorbed in the fissionable material, hence produce fission, and ν is the number of fast neutrons produced per fission. Let $K(|\underline{r} - \underline{r}'|)$ be the distribution function of the displacement occurring between the production of a fast neutron and its thermalization. Then the rate of production of thermal neutrons will be

$$m(\underline{r}) = \nu c \int d\underline{r}'\, n(\underline{r}')\, K(|\underline{r} - \underline{r}'|)$$

where the integration is carried over the core. Denoting by $g(\underline{r})$ as before that part of the thermal-neutron distribution, $n(\underline{r})$, which lies in the core, we may write this equation as

$$m(\underline{r}) = \nu c \int d\underline{r}'\, g(\underline{r}')\, K(|\underline{r} - \underline{r}'|) \tag{3.21}$$

where the integration may now be carried over all space. The diffusion distance is smaller in the core than in the water tamper because of the additional absorption of the fissionable material. The mean square diffusion distance is diminished in proportion to the absorption rate, i.e. by a factor $1 - c$ where c is as before the ratio of the absorption rate for the fissionable material to the total absorption rate. Thus $1 - c$ is the fraction absorbed by hydrogen. The distribution function of the displacement by thermal diffusion in the water tamper before capture, normalized to

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[redacted]

unity, is

$$\frac{6}{4\pi r_o^2} \; \frac{e^{-\sqrt{6}\,|r - r'|/r_o}}{|r - r'|} = Y(|r - r'|)$$

where r_o^2 is the three-dimensional mean square displacement. If for the moment we neglect the additional absorption by the fissionable material in the core then

$$n(\underline{r}) = c \int d\underline{r}' \; m(\underline{r}') \; Y(|\underline{r} - \underline{r}'|) \; 1/(1 - c)$$

where the Yukawa kernel is defined for a value of r_o appropriate for pure water. The factor $1/(1 - c)$ arises from the fact that the mean life-time in the core material was taken as the unit of time. Therefore the mean lifetime in water is $1/(1 - c)$, hence also the ratio of $n(\underline{r})$ to $m(\underline{r})$ if both are constant. The additional absorption by the fissionable material in the core results in the disappearance of neutrons at a rate $c\, n(\underline{r})$ in the core, hence at a rate $c\, g(\underline{r})$. This disappearance of neutrons may be treated as a negative source of neutrons corresponding to the positive source $m(\underline{r})$, hence results in a diminution of the neutron density, $n(\underline{r})$, by an amount $-\delta n(\underline{r})$ where

$$-\delta n(\underline{r}) = 1/(1 - c) \int d\underline{r}' \; Y(|\underline{r} - \underline{r}'|) \; c\, g(\underline{r}')$$

Thus the true neutron density is

$$n(\underline{r}) = 1/(1 - c) \int d\underline{r}' \; Y(|\underline{r} - \underline{r}'|) \left[m(\underline{r}') - c\, g(\underline{r}') \right] \qquad (3.22)$$

[redacted]

 [redacted]

If we now substitute in this equation the expression for $m(\underline{r})$ given by equation (3.21) we get a displacement integral equation for $n(\underline{r})$

$$n(\underline{r}) = 1/(1-c)\int d\underline{r}'\, Y(|\underline{r}-\underline{r}'|)\left[\nu c\int d\underline{r}''\, K(|\underline{r}''-\underline{r}'|)\, g(\underline{r}'') - cg(\underline{r}')\right] \tag{3.23}$$

which we may write as

$$n(\underline{r}) = \int d\underline{r}'\, H(|\underline{r}-\underline{r}'|)\, g(\underline{r}') \tag{3.24}$$

where

$$H(|\underline{r}-\underline{r}'|) = 1/(1-c)\left[\nu c\int d\underline{r}''\, Y(|\underline{r}-\underline{r}''|)\, K(|\underline{r}''-\underline{r}'|) - c\, Y(|\underline{r}-\underline{r}'|)\right] \tag{3.25}$$

The "slowing kernel" $K(|\underline{r}-\underline{r}'|)$ will presumably have somewhat the character of the Milne kernel and somewhat the character of a Gauss kernel since the first paths are comparable with the total displacement. The value of the kernel for large argument will be determined primarily by the first paths giving the kernel an exponential tail as in the Milne kernel. The quadratic singularity of the Milne kernel of the first path will, however, be smoothed over by the later small displacements. A very plausible approximation which combines these features is the result of compounding a Milne with a Gauss kernel,

$$K(|\underline{r}-\underline{r}'|) = \int d\underline{r}''\, K_{Milne}(|\underline{r}-\underline{r}''|)\, K_{Gauss}(|\underline{r}''-\underline{r}'|) \tag{3.26}$$

[redacted]

The Milne and Gauss kernels are determined by a specification of their mean square displacements. The total mean square displacement is the sum of these two. The total mean square displacement can be determined in a number of different ways. It has been measured experimentally and can be calculated by a variety of simple arguments. One such simple argument will be given below in this section.

Fermi Formula for Mean r^2 in Hydrogenous Material

The best calculation is that by Fermi which takes into account correctly the correlation in direction and magnitude between successive paths and the effect of nonhydrogenous scatterers. The Fermi formula is correctly as follows:

$$
\begin{aligned}
\overline{r^2} = {} & 2\lambda^2(0)\left[1+\rho(0)\right] + 2\lambda^2(a)\left[1+\rho(a)\right] \\
& + 2\lambda(0)\,\lambda(a)\, e^{-\left[a/2 + \int_0^a dx\,\rho(x)/\left(1+\rho(x)\right)\right]} \\
& + 2\int_0^a \lambda^2(x)\left[1+\rho(x)\right]dx \\
& + 2\lambda(0)\int_0^a \lambda(x)\, e^{-x/2}\, e^{-\int_0^x \rho(\zeta)d\zeta/\left(1+\rho(\zeta)\right)}\,dx \\
& + 2\lambda(a)\int_0^a \lambda(x)\, e^{-\left[(a-x)/2 + \int_x^a \rho(\zeta)\,d\zeta/\left(1+\rho(\zeta)\right)\right]}dx \\
& + 2\int_0^a \lambda(x)\int_0^{a-x} \lambda(x+\xi)\, e^{-\left[\xi/2 + \int_0^\xi d\zeta\,\rho(x+\zeta)/\left(1+\rho(x+\zeta)\right)\right]}d\xi\,dx
\end{aligned}
$$

 [redacted]

This formula was incorrectly given in the original article[3], presumably by reason of typographical errors, and cited in the incorrect form in the article of G. Horvay[4]. The difference between the two forms of the formula does not lead to a great difference in the mean r^2.

Two different methods have been used to determine the proper distribution of the total mean square displacement between the Milne and Gauss parts of the kernel. The first method, used by Christy, is to take from the experimental measurements the decay rate of the tail of the slowing distribution at large distances. This gives the coefficient entering in the Milne part of the kernel. The second method is to distribute the mean r^2 between the two parts so as to fit correctly both the mean square and mean fourth power displacements. These two moments can be calculated by the following simple theory:

Moments of the Distribution for Slowing in Hydrogen

The scattering cross section of hydrogen follows a fairly good $1/v$ law from a few tens of kilovolts up to two or three million volts. The failure of the $1/v$ law at small energy is easily taken into account since there the paths are short, hence contribute only to the Gauss part of the kernel. The angular distribution of hydrogen scattering follows a cosine

3) E. Fermi, Ric. Scient., 7, (2), 13 (1936).

4) G. Horvay, Phys. Rev. 50, 897 (1936).

[redacted]

 [redacted]

law for forward scattering. The fraction scattered into a range $d\mu$ of the cosine, μ, of the scattering angle is $2\mu d\mu$ for $0 \leq \mu \leq 1$. If the cosine of the scattering angle is μ, then the energy is reduced by a factor μ^2 and the velocity by a factor μ. If we assume that this angle and energy distribution and the $1/v$ law hold down to zero energy, then the distribution of displacements in slowing to zero energy is convergent. Moreover, because of the $1/v$ law, this distribution will vary with initial energy only by a scale factor. The linear scale of the distribution will be proportional to the initial velocity. This fact permits the determination of the first few moments of the distribution by a recursion argument. We choose for the unit of length the initial mean free path. In each stage of the argument the total displacement will be represented as the sum of the first path and the resultant of all successive paths, denoted by $\underline{r}_1$ and $\underline{r}_2$ respectively. The various moments of this resultant of the second and succeeding paths, $\underline{r}_2$, are related to the corresponding moments of the distribution of the total displacement, $\underline{r} = \underline{r}_1 + \underline{r}_2$, by the scaling relationship. The direction of the first path will be taken to be along the x-axis. The mean displacement, $\bar{r}_x$, (the bar here represents the average value) is the sum of the mean length of the first path -- by definition unity -- and mean x-component, $\bar{r}_{2x}$, of the remaining displacement, $\underline{r}_2$, the first path of which has the mean length μ. The mean remaining displacement, $\bar{\underline{r}}_2$, is oriented at an angle of cosine μ to the x-axis and is of magnitude μ times the overall mean displacement, $\bar{r}_x$; hence has an x-component of $\mu^2\bar{r}_x$. Averaging over the probability distribution for μ gives

[redacted]

[redacted]

$$\bar{r}_x = \bar{r}_1 + \bar{r}_{2x} = 1 + \int_0^1 2\mu\, d\mu\, (\mu^2 \bar{r}_x) = 1 + \frac{1}{2}\bar{r}_x \tag{3.28}$$

$$\therefore\ \bar{r}_x = 2, \qquad \bar{r}_{2x} = 1$$

In the same way we calculate the mean square displacement,

$$\overline{r^2} = \overline{r_1^2} + 2\bar{r}_1\bar{r}_{2x} + \overline{r_2^2}$$

Since the length of the first path is exponentially distributed, its mean square is 2. In the cross term the averaging over the lengths of the first path and the x-component of the remaining paths can be done separately since they are independently distributed. This term has therefore the mean value 2. r_2^2 has, for an angle cosine μ, a mean value of $\mu^2\overline{r^2}$.

$$\overline{r^2} = 2 + 2 + \overline{r^2}\int_0^1 2\mu^3\, d\mu = 4 + \frac{1}{2}\overline{r^2}$$

$$\overline{r^2} = 8, \qquad \overline{r_2^2} = 4 \tag{3.29}$$

It is seen from this that after one collision the remaining mean square displacement is reduced by half. The effect of the first path is therefore clearly not negligible. The series of contributions to the mean square displacement is rapidly convergent, thus in replacing the part of the slowing below, say, 30 Kv. by an appropriate Gaussian spread the subtraction of the effect of this part of the slowing is relatively unimportant.

The mean square x-component, and the fourth-power displacement are

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calculated in the same way,

$$\overline{r_x^2} = \overline{r_1^2} + 2\overline{r_1}\,\overline{r_{2x}} + \overline{r_{2x}^2}$$

$\overline{r_{2x}^2}$ is, for a definite μ, equal to μ^2 times the mean square r-component in a direction with an angle-cosine μ with respect to the x-axis. Take this direction in the x-y plane.

$$(\overline{r_{2x}^2})_\mu = \mu^2\,\overline{(\mu r_x + \sqrt{1-\mu^2}\,r_y)^2}$$

$$= \mu^2\left[\mu^2\overline{r_x^2} + (1-\mu^2)\left(\frac{\overline{r^2}-\overline{r_x^2}}{2}\right)\right]$$

$$= \frac{1}{2}(\mu^2-\mu^4)\,\overline{r^2} + \frac{1}{2}(3\mu^4-\mu^2)\,\overline{r_x^2}$$

$$= (\mu^2-\mu^4) + \frac{1}{2}(3\mu^4-\mu^2)\,\overline{r_x^2}$$

$$\overline{r_x^2} = 2 + 2 + \int_0^1 2\mu\,d\mu\left[(\mu^2-\mu^4) + \frac{1}{2}(3\mu^4-\mu^2)\,\overline{r_x^2}\right]$$

$$\overline{r_x^2} = 4 + \left(\frac{2}{4}-\frac{2}{6}\right) + \left(\frac{3}{6}-\frac{1}{4}\right)\overline{r_x^2}$$

$$\overline{r_x^2} = \frac{56}{9}, \qquad \overline{r_{2x}^2} = \frac{20}{9}$$

$$\overline{r_x r^2} = \overline{(r_1+r_{2x})(r_1^2+2r_1r_{2x}+r_2^2)}$$

$$= \overline{r_1^3} + 3\overline{r_1^2}\,\overline{r_{2x}} + 2\overline{r_1}\,\overline{r_{2x}^2} + \overline{r_1}\,\overline{r_2^2} + \overline{r_{2x}r_2^2}$$

$$= 6 + 3\cdot 2\cdot 1 + 2\cdot 1\cdot\frac{20}{9} + 1\cdot 4 + \int_0^1 2\mu\,d\mu\;\mu^3\cdot\mu\;\overline{r_x r^2}$$

rescaling x-component

[redacted]

[redacted]

$$\overline{r_x r^2} = \frac{92}{3}, \qquad \overline{r_{2x} r_2^2} = \frac{92}{9}$$

$$\overline{r^4} = \overline{(r_1^2 + 2r_1 r_{2x} + r_2^2)^2}$$

$$= \overline{r_1^4} + 4\overline{r_1^2}\,\overline{r_{2x}^2} + 4\overline{r_1^3}\,\overline{r_{2x}} + 4\overline{r_1}\,\overline{r_{2x} r_2^2} + 2\overline{r_1^2}\,\overline{r_2^2} + \overline{r_2^4}$$

$$= 24 + 4\cdot 2\cdot \frac{20}{9} + 4\cdot 6\cdot 1 + 4\cdot 1\cdot \frac{92}{9} + 2\cdot 2\cdot 4 + \int_0^1 2\mu\, d\mu\, \mu^4\, \overline{r^4}$$

$$\overline{r^4} = 184$$

<u>Moments of Distribution for Slowing in Hydrogen Plus Heavy Material</u>

This calculation of the second and fourth moments does not take into account the effect of scattering by nuclei other than those of hydrogen. The effect of the inclusion of heavy elements will be taken into account in two successive approximations. In both cases the heavy nuclei will be considered infinitely heavy compared with the neutron, the scattering elastic and isotropic. In the first approximation the effect of the heavy material will not be taken into account after the first hydrogen collision. In this approximation we have calculated the mean square and fourth power displacements. The result of this calculation shows the ratio of the fourth moment to the square of the second moment (which ratio determines the distribution of the mean square displacement between the Milne and Gauss kernels) to be very insensitive to the concentration of heavy material assumed.

The second approximation used assumes a scattering cross section for the heavy material which is independent of the energy. The heavy

[redacted]

material scattering is then taken into account in all stages of the slowing. The treatment in this approximation is considerably more laborious than the preceding approximation and has therefore been carried only as far as the evaluation of the second moment. Since the dependence of the moment ratio on the concentration of heavy material is so slight in the first approximation it was considered sufficient to use this first approximation in evaluating the moment ratio while using the result of the second approximation method in determining the mean square displacement. The second moment is more accurately determined by the Fermi formula, but this formula is very inconvenient to use as it involves a number of numerical integrals over experimental curves. The mean square displacement in slowing in water has been calculated by P. Morrison[5] using the Fermi formula. However we do not now know whether the correct form of the Fermi formula or the incorrect published form was used. In the absence of this knowledge the result of the second-approximation method is the most convenient formula which is of sufficient accuracy.

In the first approximation the scattering probability for hydrogen is again taken to be one per unit length for the initial energy and varying as $1/v$ as before. The scattering probability for the heavy material is a per unit length for the initial energy and zero for smaller energies. As before we divide the overall displacement into the first path and subsequent

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5) (CF-631)

[redacted]

paths.

$$\overline{r_x} = \overline{r_1} + \overline{r_{2x}}.$$

The mean length of the first path is now $1/(1 + a)$. The first collision can either be a hydrogen collision, with probability $1/(1 + a)$, or a heavy material collision, with probability $a/(1 + a)$. If the first collision is with heavy material, then, since this collision is isotropic, the mean x-component of the remaining paths is zero. If the first collision is with hydrogen, then since the heavy-material cross section is to be neglected thereafter the mean x-component of the remaining paths is just that given by the preceding calculation (3.28), i.e. 1.

$$\overline{r_x} = 2/(1 + a)$$

similarly

$$\overline{r^2} = \overline{r_1^2} + 2\overline{r_1}\,\overline{r_{2x}} + \overline{r_2^2}$$

$$= \frac{2}{(1 + a)^2} + 2 \cdot \frac{1}{1 + a} \cdot \frac{1}{1 + a} + \overline{r_2^2}$$

$\overline{r_2^2}$ may be divided into two parts. If the first collision is with heavy material, with probability $a/(1 + a)$, then $\overline{r_2^2} = \overline{r^2}$. If the collision is with hydrogen, with probability $1/(1 + a)$, then $\overline{r_2^2}$ is, as before, 4.

$$\overline{r^2} = \frac{4}{(1 + a)^2} + \frac{a}{1 + a}\overline{r^2} + \frac{4}{1 + a}$$

$$\overline{r^2} = \frac{8 + 4a}{1 + a}$$

[redacted]

[redacted]

The rest of the calculation proceeds in the same way. The result obtained is

$$\overline{r^4} \quad \frac{1656 + 3176a + 2264a^2 + 552a^3}{9(1 + a)^3}$$

The ratio, $\overline{r^4}/\overline{r^2}^2$, varies only slightly with a as seen in the following table.

a	$\overline{r^4}/\overline{r^2}^2$
0	2.875
.2	2.853
.4	2.862
.6	2.887
.8	2.917
1.0	2.951

In the second approximation the effect of the heavy material scattering is taken into account in all stages. After a hydrogen collision the energy is reduced, hence the hydrogen cross section is increased. Since the heavy material cross section is not increased the scale-factor recursion no longer holds in the original simple form. If the velocity is reduced by a factor μ we may consider the distribution of remaining paths scaled down by a factor μ if also the heavy material concentration, a, is reduced by the same factor. Thus

[redacted]

[redacted]

$$\bar{r}_x(a) = \bar{r}_1(a) + \bar{r}_{2x}(a)$$

$$= 1/(1+a) + 1/(1+a) \cdot \int_0^1 2\,\mu d\mu\, \mu^2\, \bar{r}_x(\mu a)$$

$$\bar{r}_x(a) = \sum_{s=0}^{\infty} b_s\, a^s$$

$$(1+a)\bar{r}_x(a) = b_0 + \sum_{s=1}^{\infty} a^s\,(b_s + b_{s-1})$$

$$= 1 + \sum_{s=0}^{\infty} b_s\, a^s \int_0^1 2\, d\mu\, \mu^{3+s}$$

$$= 1 + \sum_{s=0}^{\infty} b_s\, a^s \left(\frac{2}{4+s}\right)$$

$$b_0 = 2$$

$$b_s = -\frac{4+s}{2+s}\, b_{s-1} \qquad s \geq 1$$

$$b_s = (-)^s\, \frac{1}{6}\, \frac{(4+s)!}{(2+s)!} = (-)^s\, \frac{1}{6}\,(4+s)(3+s)$$

$$\bar{r}_x(a) = \frac{1}{6a^2}\, \frac{\partial^2}{\partial a^2}\left(\frac{a^4}{1+a}\right) = \frac{6+8a+3a^2}{3(1+a)^3}$$

$$\overline{r^2}(a) = \overline{r_1}^2 + 2\bar{r}_1\bar{r}_{2x} + \overline{r_2}^2$$

$$= \frac{2}{(1+a)^2} + 2\cdot\frac{1}{1+a}\cdot\left(\frac{6+8a+3a^2}{3(1+a)^3} - \frac{1}{1+a}\right) + \frac{a}{1+a}\,\overline{r^2}(a)$$

$$+ \frac{1}{1+a}\int_0^1 2\mu d\mu\, \mu^2\, \overline{r^2}\,(\mu a)$$

[redacted]

$$\overline{r^2}(a) = \int_0^1 2\,\mu^3 d\mu\, \overline{r^2}(\mu a) = 2\overline{r_n^2}(a)$$

$$\overline{r^2}(a) = \sum_{s=0}^{\infty} C_s\, a^s, \qquad \int_0^1 2\,\mu^3 d\mu\, \overline{r^2}(a) = \sum_{s=0}^{\infty} C_s\, a^s \frac{2}{4+s}$$

$$\frac{2+s}{4+s}\, C_s = 2b_s$$

$$C_s = (-)^s\, \frac{1}{3}\, \frac{(4+s)^2\,(3+s)}{(2+s)}$$

$$\overline{r^2}(a) = \frac{1}{3a^2}\, \frac{d^2}{da^2}\left[a\, \frac{d}{da}\left(a^2 \int_0^a \frac{x}{1+x}\, dx\right)\right]$$

$$= -\frac{4\,\ln(1+a)}{3a^2} + \frac{4 + 34a + 46a^2 + 18a^3}{3a\,(1+a)^3}$$

A graph of this function is presented in Fig. XIII.

The compounding of the Milne and Gauss parts of the Christy kernel is best done with the three-dimensional Fourier transform.

$$\overline{K}(\underline{k}) = \int d\underline{r}\; e^{i\underline{k}\cdot\underline{r}}\, K(\underline{r})$$

Since $K(\underline{r})$ is normalized and symmetric

$$\overline{K}(k) = \int_0^\infty 4\pi r^2\, dr\, \frac{\sin kr}{kr}\, K(r)$$

$$= \int_0^\infty 4\pi r^2 K(r)\left(1 - \frac{k^2 r^2}{3!} + \frac{k^4 r^4}{5!} - \ldots\ldots\right)$$

$$= 1 - \frac{k^2}{3!}\,\overline{r^2} + \frac{k^4}{5!}\,\overline{r^4} - \;\ldots\ldots\ldots \qquad (3.30)$$

[redacted]

Mindex Theory

In compounding two kernels as in (3.26) we find the distribution of the resultant of two independently distributed displacements. The Fourier transform of the distribution of this resultant is the product of the Fourier transforms of the distributions of the two independent displacements. Denoting the sedond and fourth moments of the first distribution by a and b, of the second by a' and b', and of the resultant distribution by A and B,

$$1 - \frac{k^2A}{6} + \frac{k^4B}{120} \dots = \left(1 - \frac{k^2a}{6} + \frac{k^4b}{120} \dots\right)\left(1 - \frac{k^2a'}{6} + \frac{k^2b'}{120} \dots\right)$$

$$A = a + a'$$

$$B = b + b' + \frac{10}{3} a a'$$

Thus

$$\left(B - \frac{5}{3} A^2\right) = \left(b - \frac{5}{3} a^2\right) + \left(b' - \frac{5}{3} a'^2\right)$$

Define

$$M = \sqrt{\frac{3B}{5A^2} - 1}\ , \quad \text{and similarly } m \text{ and } m'.$$

The expression M, known as the mindex[6], depends only on the character of the distribution and not on the scale of size.

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6) Terminology owing to R. Feynman. [redacted]

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 [redacted]

$$(MA)^2 = (ma)^2 + (m'a')^2 \tag{3.31}$$

For a Gauss function, $M = 0$[7]. For a Milne kernel, $M = 1.61245$, for a Yukawa kernel, $M = 1.0000$. Taking $D/A^2 = 2.9$, as indicated by the result of the first approximation method for slowing in water, gives $M = .860$. Writing (3.31) in the form

$$M^2 = \frac{m^2a^2 + m'^2a'^2}{(a + a')^2}$$

shows that the magnitude of M^2 is diminished by the process of compounding distributions. This is consistent with the fact that the limiting form of compounded distributions is Gaussian, with zero mindex.

Constants in the Christy Kernel

In the present case one of the two functions is Gaussian, hence $m' = 0$.

$$M^2/m^2 = \left[a/a + a'\right]^2$$

$$a/(a + a') = .860/1.612 = .534$$

One-sixth of the second moment is known as the "age" of a neutron distribution, thus the age, $A/6$, must be distributed about equally between the Milne and

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7) This is a partial statement of the more general property of stability of Gauss distributions under compounding.

[redacted]

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Gauss parts of the kernel. This [illegible] would indicate that the Gauss age should be about 13 percent less than the Milne age. However this treatment underestimates the Gauss age because of the use of the 1/v law below 30 Kv. Since in the low-energy range the paths are small, this region contributes only to the Gauss part of the kernel. The increase in the total age produced by correcting the low-energy cross sections must be added only to the Gauss age. The effect of this correction is to make the Gauss age ten or fifteen percent greater than the age of Milne.

Critical Mass of Water-Boiler

The characteristic equation curve has been calculated for the water-boiler kernel, (3.25), for two sets of coefficients: one used by R. Christy in his calculation of the critical mass of the water-tamped water-boiler, the other corrected by the use of the results of this calculation (see Fig. XIV). Christy's coefficients give the Gauss age about 10 percent less than the Milne age in the slowing kernel. Correcting this ratio makes a negligible change in the characteristic equation curve in the neighborhood of the concentration giving the minimum mass. The extrapolated end-point was calculated with the Christy coefficients in this neighborhood and the results are shown in Fig. XV. The minimum critical mass, .580 Kg (calculated, however, with $\nu = 2.2$), agrees with Christy's result. This mass will probably be increased by about 30 percent if $\nu = 2.0$.

The change in the coefficients in the slowing kernel produces a small but appreciable change in the characteristic equation curve away from the optimum region. This would produce some change in the extrapolated end-

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point. However, it is believed that the resulting change in the critical mass would not be appreciable in comparison with the uncertainty in the constants.

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CHAPTER IV. OTHER METHODS OF SOLUTION AND RELATED PROBLEMS

In this chapter we propose to discuss the various methods of treating integral equations, other than the end-point method, which have proved of value in the present work. The two most important of these are the variation method and the iterative numerical solution. The variation method is for most problems the most accurate method of treatment now in use. It is quite flexible and can be applied to tamped and untamped problems of almost any shape. However, the difficulty of evaluating the integrals involved increases rapidly with increasing complexity of shape. For this reason it has been applied to only a few examples of each of several types of geometry, spheres, slabs, cylinders, and rectangular solids. For this reason it does not seem particularly promising for the solution of problems of greater complexity than those already treated.

Numerical Method

The numerical method is the simplest and probably the most flexible method used to date. Although it is the first method used which gave reasonably accurate results it is the least well developed method of solution in use. In its present rudimentary form it can be applied to problems of considerable geometric complexity only with an enormous expenditure of computational labor. It is hoped that further refinement of the techniques of application of this method will make practicable the solution of integral

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equations of types already treated, for much more complicated geometry. In applying this method to the solution of an integral equation of the form (1.1) a plausible guess is made as to the shape of the function $n(\underset{\sim}{x})$, say $n_0(\underset{\sim}{x})$. This guessed function is inserted in the right side of the equation and the integration carried out numerically. The resulting function, $n_1(\underset{\sim}{x})$, is again integrated numerically to give a next approximation. $n_2(\underset{\sim}{x})$. This iteration process is continued until the successive approximants differ only by a multiplicative factor. This factor is the highest eigenvalue c. Since $n_0(x)$ can be represented as a superposition of the solutions of the integral equation for various eigenvalues the rapidity of convergence will be determined primarily by the ratio of the highest to the second highest eigenvalue. In the problems so far treated this ratio has been of the order of 1.5 or 2. Four or five iterations are usually sufficient to give a value of c which is stable to a few tenths of one percent. If the numerical integrations have been carried out properly this will be the accuracy of the solution. This method has so far been applied only to slabs and spheres where the integral equation can be reduced to one in one dimension with a displacement kernel. The iteration process can then be set up in a very simple form and can be carried out by relatively untrained computers.

The iterative numerical solution is of particular value when not only the eigenvalue but also the eigenfunction is desired. A number of solutions of spherical problems for the Milne kernel have been obtained, however, with fairly crude integration techniques. (Trapezoidal integration is used except for the singularity of the kernel which is treated to give

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the correct integral with a constant or linear cofactor.) A few of the curves obtained are given in Fig. XVI. A more careful solution for a Gaussian kernel was obtained for three sizes of "untamped"* spheres, a = 1, 1.6 and 2. The resulting curves are presented in Fig. XVII.

In the numerical treqtment of spherical and slab problems the displacement character of the one dimensional kernel permits a very simple integration technique. The set of numerical entries representing the kernel is written on a strip of paper. The numerical entries of the successive trial functions are written on a parallel strip. The integration is then performed by summing the products of adjacent pairs of numbers, the value of x being determined by the position of the center point of the kernel. A different value of x is gotten merely by displacing the strips. The numerical treatment of problems for which the one dimensional form of the kernel has not this displacement property, e.g. the infinite cylinder, would require making a separate kernel strip for each value of x. The iteration method is, of course, not restricted to problems which can be reduced to one-dimensional integral equations. However, the application of the method would become prohibitively laborious if the number of entries for integration were large. It is hoped that the use of powerful integration methods and the judicious choice of representative regions will give reasonably accurate results for a moderate number of entries. Preliminary investigations are now being carried out. These indicate that it will be possible to use this method for a limited number of problems of complicated geometry. The results of these may be used to validate simpler recipes which can be applied to many cases. [redacted]

* This describes approximately a water tamped water boiler. It is "untamped" in the sense that the integration is carried only over the core.

[redacted]

Variation Method

The variation method of solution of integral equations here dealt with is essentially an application of the familiar Ritz method of solution of differential equations. The kernel is here a fairly smooth function extending over the entire region of integration, thus small errors in the shape of the trial function are much less important than if the kernel involved only a delta function and its derivatives. Thus, for example, in the treatment of the untamped sphere a parabolic trial function (one free parameter) proved far more accurate than necessary. The constant trial function, however, is not sufficiently accurate for the sphere, and presumably would be still more unsatisfactory for more complicated shapes.

In the variation method the integral equation

$$n(\underline{r}) = c\int d\underline{r}'\, K(|\underline{r} - \underline{r}'|)\, F(\underline{r}')\, n(\underline{r}')$$

is expressed by the variation equation

$$\delta(I/N) = 0, \qquad I/N = 1/c \tag{4.1}$$

where

$$I = \int d\underline{r}\, d\underline{r}'\, K(|\underline{r} - \underline{r}'|)\, F(\underline{r})\, F(\underline{r}')\, n(\underline{r})\, n(\underline{r}')$$

$$N = \int d\underline{r}\, F(\underline{r})\, n^2(\underline{r})$$

and δ denotes variation of $n(\underline{r})$. Usually only the smallest eigenvalue, c, is of interest, hence the greatest maximum of I/N. This method has

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been most extensively used with the Milne kernel. Some of the results so obtained and a comparison with the end-point method results are given in Figs. XVIII, A, B and C. For the simplest shapes the calculation was usually made with a constant and with a parabolic trial function. The constant trial function result is given and also the minimum obtained by varying the one effective free parameter in the parabolic trial function. The non-cubical rectangular solids were calculated with two free parameters and the finite cylinders with three. In all cases the end-point method was used to extend the few variation results to other sizes.

Integral Boundary Condition Method

Another analytic approximation method which was used in the early stages of this investigation, the integral-boundary-condition method, gives reasonably accurate results. In this method the asymptotic interior solution is used throughout each region of constant F. The phases of these interior solutions are determined by the requirement that at the boundaries the integral equation be satisfied (cf. LA-5). Here also the approximation of assuming the several regions infinitely thick is of value. For example, if we apply this boundary condition to an untamped surface with the Milne kernel we have,

$$n(x) = \sin k_o(x + x_o)$$

$$\sin k_o x_o = \frac{c}{2} \int dx\, E(x) \sin k_o(x + x_o)$$

$$= \frac{c \sin k_o x_o}{2} \frac{\tan^{-1} k_o}{k_o} + c \cos k_o x_o \frac{\ln (1 + k_o^2)}{4k_o}$$

[redacted]

[redacted]

This, with the characteristic equation $(c/k_o) \tan^{-1} k_o = 1$, gives

$$\tan k_o x_o = \frac{1}{2} \frac{\ln(1 + k_o^2)}{\tan^{-1} k_o}$$

A comparison of the value of x_o so obtained with the extrapolated end-point solution is presented in this table

k_o	x_o	
	Integral Bound. Cond.	Extrapolated End-Point
0	.5	.7104
.5	.472	.6590
1.0	.416	.5584
1.5	.360	.4668
2.0	.314	.3954

and shows this solution to be reasonably accurate for the interesting range of k_o and most inaccurate for small k_o for which such inaccuracy is less significant. This method is more accurate in tamped problems where the deviations from the asymptotic solution are smaller. The only advantage of this method over the extrapolated end-point method is the ease with which it can be applied to new kernels. The integrals involved can usually be evaluated analytically or by an easy numerical solution.

[redacted]

Albedo of an Isotropic Surface Source

A problem closely related to that of the one-boundary solution of the integral equation is the albedo problem. Here the inhomogeneous solution of the same integral equation is sought, the inhomogeneous term being a surface flux of incident neutrons distributed in angle in a specified way. The general case, i.e. for an arbitrary angular distribution of incident neutrons, has been treated exactly by Halpern, Lueneburg and Clark[8]. Their treatment uses much the same method of analysis as the present extrapolated end-point method. It was their treatment which suggested to us this line of attack on problems of this type. There is one special case of the albedo problem which can be solved exactly by a much simpler method than this. As this method may prove of value in related problems we present it here. This special case is that for which the number of incident neutrons is uniform in angle. (This distribution is to be distinguished from that called isotropic by Halpern, Lueneburg and Clark, which has its flux uniform in solid angle.) The incident distribution treated here might be realized by irradiating with thermal neutrons a thin layer of fissionable material on the surface of the half-infinite medium whose albedo is considered. Half of the fission neutrons produced will enter the scattering medium uniformly in angle. The scattering material is assumed to produce only isotropic

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8) Phys. Rev. 53, 173 (1938) [redacted]

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elastic scattering and absorption.

We first neglect the absorption of neutrons in the scattering medium and compute the fraction of the incident neutrons returning across the boundary after any specified number of collisions. By summing this series with the appropriate powers of the reflected fraction in each collision the total albedo is obtained as a function of the ratio of scattering and absorption cross sections. We first compute the fraction returning after n collisions for the first few integers. The series suggested by this result is then verified by mathematical induction.

The fraction of the incident neutrons returning after one collision is one fourth. This follows from the fact that the incident neutrons are distributed uniformly in (the cosine of the) angle between 0 and $\pi/2$. After being once isotropically scattered, the neutrons are divided into two equal parts, those still going forward and those returning. The returning half of the neutrons are distributed in angle about the outward normal exactly as the incident neutrons were about the inward normal. Thus their distribution of x-component path length before collision (x parallel to the normal) is exactly that of the incident neutrons: i.e. their distribution in distance from the boundary after the first collision is the same as the distribution of (x-component) displacement before having another collision. Of the half of the incident neutrons returning toward the boundary after the first collision, half will cross the boundary without suffering a further collision.

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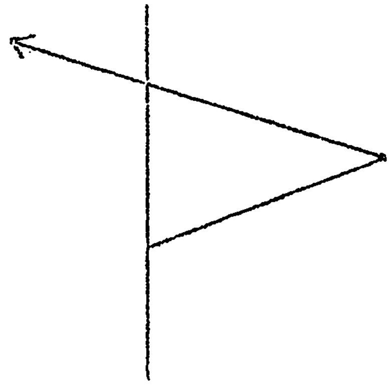

Fig. II

The fraction returning after two collisions we divide into two parts: those, f_1, still going forward after their first collision which come back after their second collision and cross the boundary without further collisions, and those, f_2, which are returning after their first collisions, have another collision before reaching the boundary and then cross the boundary without another collision.

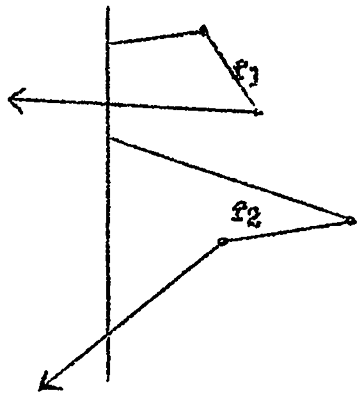

Fig. III

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The first fraction, f_1, is the probability, 1/2, that a neutron is still going forward after its first collision, times the probability, 1/2, that it go backward after the second collision, times the probability that the backward displacement be greater than the sum of the two forward displacements. The fraction, f_2, is the probability, 1/2, of going backward after the first collision, times the probability, 1/2, of going backward after the second collision, times the probability that the forward displacement be less than the sum of the two backward displacements but not less than the first of them. The sum of these is

$$\frac{1}{4}\left[P(1+2<3) + P(1>2,\ 1<2+3)\right]$$

where the first P represents the probability that the sum of two displacements be greater than a third, the second P the probability that one displacement be greater than a second but less than this second plus a third. The first P may equally well be written $P(3>2,\ 3>1+2)$, the probability that one displacement be greater than another and also greater than this other plus still another (since if it is greater than both it is greater than either one). Since the indices, 1, 2, 3, have no special significance beyond labeling the several paths all of which have the same probability distribution in length, the sum becomes

$$\frac{1}{4}\left[P(1>2,\ 1>2+3) + P(1>2,\ 1<2+3)\right]$$

$$= \frac{1}{4}P(1>2) = \frac{1}{8}$$

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since the sum of the two probabilities bracketed above is just the total probability that any one path exceed another.

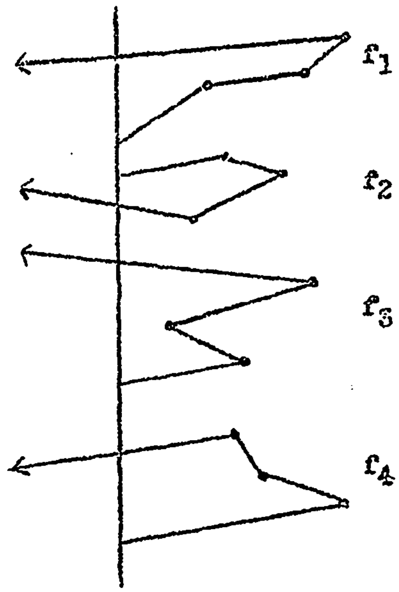

Fig. IV

Similarly the fraction of neutrons returning after three collisions is

$$\frac{1}{8}\Big\{ P(4>1+2+3) + P(3<1+2,\ 3+4>1+2) + P(2<1,\ 4+2>1+3) + P(1>2+3,\ 1<2+3+4)\Big\}$$

$$=\frac{1}{8}\Big\{\big[P(4>2+3,\ 4>1+2+3) + P(1>2+3,\ 1<2+3+4)\big] + P(3<1+2,\ 3+4>1+2) + P(2<1,\ 4+2>1+3)\Big\}$$

$$=\frac{1}{8}\Big\{\big[P(1>2+3)\big] + P(3<1+2,\ 3+4>1+2) + P(2<1,\ 4+2>1+3)\Big\}.$$

[redacted]

Here

$$P(1 > 2 + 3) = P(3 > 1 + 2) = 1 - P(3 < 1 + 2)$$

$$= 1 - P(3 < 1 + 2,\ 3 + 4 > 1 + 2) - P(3 < 1 + 2,\ 3 + 4 < 1 + 2)$$

$$P(1 > 2 + 3) + P(3 < 1 + 2,\ 3 + 4 > 1 + 2) = 1 - P(3 < 1 + 2,\ 3 + 4 < 1 + 2)$$

$$= 1 - P(3 + 4 < 1 + 2)$$

$$= 1 - \frac{1}{2} = \frac{1}{2}$$

$$P(2 < 1,\ 4 + 2 > 1 + 3) = P(2 < 1,\ 4 > 3,\ 4 - 3 > 1 - 2)$$

$$= P(2 < 1)\ P(4 > 3)\ P(|4 - 3| > |1 - 2|)$$

$$= \frac{1}{8}$$

The total fraction returning after three collisions is then

$$\frac{1}{8}\left\{\frac{1}{2} + \frac{1}{8}\right\} = \frac{5}{64}$$

Similarly calculated, the fraction returning after four collisions is 7/128.

The series of coefficients, 1/4, 1/8, 5/64, 7/128... can be written as

$$\frac{1}{4},\ \frac{1}{4}\frac{3}{6},\ \frac{1}{4}\frac{3}{6}\frac{5}{8},\ \frac{1}{4}\frac{3}{6}\frac{5}{8}\frac{7}{10},\ \ldots\ldots$$

suggesting that the fraction returning after n collisions, F_n, is $C_n^{2n+1}/2^{2n}$ where C_b^a is the number of combinations of a things taken b at a time, $a!/\left[b!\ (a - b)!\right]$.

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The fraction remaining in the scattering medium is then

$$1, \quad \frac{3}{4}, \quad \frac{3}{4}\cdot\frac{5}{6}, \quad \frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}, \quad \frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\cdot\frac{9}{10}, \quad \ldots\ldots$$

Thus after n paths (at the nth collision) the fraction remaining is $C^{2n}_{n}/2^{2n-1}$. We show that this is true for all n as follows: The series of n paths joined by the $n-1$ collisions can be represented as a series of cycles, each being an excursion into the scattering medium and a turning back. Each of the n paths except the first is equally likely to be directed into the medium or out. Similarly each except the first is as likely to be directed in the same sense as the preceding as in the opposite sense. Thus at each collision there may or may not be, with equal probability, an inversion of direction. The first, third, fifth, etc., such inversions determine successive cycles. If n is $2s$ or $2s+1$ there may be at most s cycles. The number of cycles is half of the number of inversions or half of one plus the number of inversions, whichever is integral. Thus for $n-1$ collisions separating n paths, the probability that there be s cycles is the probability that there be among the $n-1$ collisions $2s-1$ or $2s$ inversions. This probability is

$$\frac{1}{2^{n-1}}\left(C^{n-1}_{2s-1} + C^{n-1}_{2s}\right) = \frac{1}{2^{n-1}}\, C^{n}_{2s}$$

The total displacement in each cycle may be either positive or negative with a symmetric probability law just as in each path. The probability

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law is the same for each of the s cycles so long as the number of paths in the various cycles is not specified. The probability that nowhere in the s cycles does the path cross the boundary is just one half of the probability that after s separate paths there be no recrossing of the boundary. The factor 1/2 arises from the fact that in the original problem it was specified that the first path is directed into the medium. This will be true only half of the time for the first cycle.

We make the ansatz that the fraction of neutrons remaining after n paths is $C^{2n}_{n}/2^{2n-1}$ as suggested by the first few terms. Then since the first line of argument used to find this probability applies equally well to a series of n cycles (except for the factor 1/2) it must be true that the probability of remaining after n cycles is $C^{2n}_{n}/2^{2n}$. Since, however, the probability that in n paths there be s cycles is $C^{n}_{2s}/2^{n-1}$ and the probability of remaining in the medium in these s cycles is $C^{2s}_{s}/2^{2s}$ the total probability of remaining after n paths is

$$\left(1/2^{n-1}\right) \sum_{s} C^{n}_{2s}\, C^{2s}_{s}/2^{2s}$$

where the summation is carried over all integers, s, for which the two combinatorial symbols do not vanish. This expression will be recognized as the constant term in the double binomial expansion of

[redacted]

$$\left(1/2^{n-1}\right)\left(1+\frac{x+1/x}{2}\right)^{n}$$

$$=\left(1/2^{n-1}\right)\left(\frac{\sqrt{x}}{\sqrt{2}}+\frac{\sqrt{1/x}}{\sqrt{2}}\right)^{2n}$$

$$=\left(1/2^{2n-1}\right)\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^{2n}$$

of which the constant term is $C_n^{2n}/2^{2n-1}$. Thus if the ansatz is true for integers up to n, it must also be true for integers up to $2n$. Since it is true for the first few values of n it is therefore true for all n.

Having now the fraction of incident neutrons returning after n collisions in the absence of any absorption, we compute the albedo as a function of the amount of absorption by multiplying each such fraction by the appropriate power of the reflectivity, s, i.e. the fraction scattered at each collision, and summing on n. This gives for the albedo as a function of the reflectivity, $A(s)$,

$$A(s) = \frac{2-2\sqrt{1-s}}{s} - 1$$

This result is consistent with the asymptotic formula derived by Fermi for the case, $1-s \ll 1$, $A(s) \cong 1 - 2\sqrt{1-s}$. A graph of the albedo for "isotropic" incidence is given in Fig. XIX.

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Detonation Probability for Hypercritical Distributions

Another related problem is that of the detonation probability for slightly hypercritical distributions. For such a distribution of material an initial distribution of neutrons is certain to increase in number exponentially to the point of explosive expansion only if the initial number is so great that statistical fluctuation can be neglected. A question of interest, therefore, is the probability that a single neutron introduced into the distribution of material composing the gadget in some random way lead to "ignition". For simplicity we assume that the "random way" in which the single neutron is introduced have a probability distribution which is of the same shape as the spatial dependence of the hypercritical solution of the integral equation. The extent to which the distribution of material is hypercritical will be defined by the assumption that the probability that a neutron in the gadget produce a fission process (thus giving two neutrons) is p, the probability that it escape or be captured without producing fission is $q = 1 - p$. Then in each generation, so defined, the number of neutrons increases in the mean by a factor $2p = 1 + \epsilon$. This excess, ϵ, is taken as the measure of hypercriticality. We denote by P_n the probability that an initial distribution of n neutrons lead to ignition. Since the time scale is of no significance in this problem, we disregard the actual order of the processes involved and consider that first one, then another, etc., of the neutrons makes the choice between death and multiplication. With this view it becomes clear that

$$P_n = p\,P_{n+1} + q\,P_{n-1}$$

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with the condition

$$P_0 = 0, \quad P_\infty = 1$$

We look for a solution of the form $P_n = a^n$, or a linear combination of two such solutions (the difference equation is linear and homogeneous). Inserting this form for P_n gives

$$a = p\,a^2 + q$$

$$a = \frac{1 \pm \sqrt{1 - 4pq}}{2p} = \frac{1 \pm \sqrt{1 - 4p + 4p^2}}{2p}$$

$$= \frac{1 \pm (2p - 1)}{2p} = \frac{1 \pm \epsilon}{1 + \epsilon} = 1, \frac{1 - \epsilon}{1 + \epsilon}$$

$$P_n = 1^n - \left(\frac{1 - \epsilon}{1 + \epsilon}\right)^n$$

$$P_1 = \frac{2\epsilon}{1 + \epsilon}$$

Thus the ignition probability is small for slightly hypercritical distributions and increases somewhat slower than linearly for increasingly hypercritical distributions.

At a time considerably after the introduction of the single neutron the expectation value of the number of neutrons present is just the number of neutrons that would be present if the distribution had grown

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exponentially without statistical fluctuation. Since this fluctuation gives only a small probability that there be any neutrons present (for small ϵ), then the mean number present if this number is other than zero, i.e. if ignition has occurred, must be greater by a factor $(1 + \epsilon)/2\epsilon$ than the number which would be produced by a smooth non-statistical growth.

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CHAPTER V. EXTENSION OF THE EXTRAPOLATED END-POINT METHOD TO OTHER SHAPES

The end-point method gives a rigorous solution to displacement integral equations where the range of integration covers all space to one side of a plane boundary. It was shown in Chapter II that this solution for a half-infinite medium can by used to supply a recipe for the solution of slab and sphere problems which is correct in the limit of large thickness or diameter and which should be of sufficient accuracy throughout the interesting range of sizes. A comparison with the variation-method results justifies this expectation. There are many problems of interest in the present work involving considerably more complicated shapes. It will be shown in this chapter that the end-point method can be applied with reasonable accuracy to many such problems, even where no simple argument can be given to justify the accuracy of the approximation. In a few problems of more complicated shape both the variation and end-point methods have been applied. The close agreement of the results of the two calculations is taken as evidence that the end-point method can safely be used in still more complicated cases, for which the variation method can be used only with a prohibitive amount of labor.

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§1. THE INFINITE CYLINDER

The radial dependence of the asymptotic solution in the interior of an infinitely long cylinder is $J_0(k_0r)$. The true solution will drop below this asymptotic solution near the surface but will not reach zero. The edge of the cylinder will therefore occur for an r somewhat smaller than the first root of $J_0(k_0r)$, i.e. $2.4048/k_0$. The amount by which this first root exceeds the radius may be called the "cylindrical end-point". It has not so far proved possible to identify the cylinder problem with some corresponding plane problem. However, for large radius, where the curvature is negligible, this end-point must have the same F-dependence as for a plane surface. Moreover it is seen from the analysis of Chapter II that curving the surface in both directions, i.e. in replacing the plane by a spherical surface, the end-point distance is not changed. This suggests the hypothesis that in introducing a curvature in one direction, i.e. in replacing the plane by a cylindrical surface, the end-point distance will still not be greatly changed. We therefore calculated critical radii for infinite cylinders of F-values by taking the radius less than the first root of the J_0 by the same extrapolated end-point distance as that used for the slab and sphere. The critical radius for a few values of F has also been calculated by the variation method (D. R. Inglis, LA-26). Both results are presented in Fig. XVIII-A. It may be seen from this figure that the discrepancy, if any, is less than the accuracy of the variation calculation. This verification of the hypothesis used in this recipe for the end-point solution of cylindrical problems extends throughout the useful range of radii.

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The corresponding recipe for the tamped infinite cylinder would be as follows: The asymptotic solutions in the core and tamper are the Bessel-function solutions for the values of k_o fixed by the characteristic equation. If the tamper is finite the phase of the asymptotic solution in the tamper is fixed by putting its first root at the extrapolated end-point distance beyond the boundary. If the tamper is infinite, regularity at infinity determines the phase. The boundary condition at the core-tamper interface requires that the logarithmic derivatives for the core and tamper asymptotic solutions agree at a radius which is less than the actual core radius by the amount Δx_o. Both the end-point, x_o, and the discrepancy term Δx_o are to be taken from the graphs calculated for plane problems.

No sufficiently accurate variation solutions have so far been performed for tamped infinite cylinders thus no check is available on the accuracy of this recipe. However, because of the close check for untamped cylinders we are confident that this recipe is as accurate as is necessary.

2. FINITE CYLINDERS

Untamped Cylinders

The success of the extension of the end-point method to infinite cylinders encouraged the attempt to find a similar recipe for untamped finite cylinders. The following recipe was tried: For a definite value of F the interior solution is taken as $\cos k_1 z \, J_o(k_2 r)$ where z is distance from the center, parallel to the axis. $k_1^2 + k_2^2 = k_o^2$ where k_o is determined by the characteristic equation. The half-length of the cylinder

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is taken less than the first root of $\cos k_1 z$ by the amount x_0, similarly the radius is taken less than the first root of $J_0(k_2 r)$ by the amount x_0, where x_0 is the extrapolated end-point distance calculated for plane boundaries in the absence of transverse waves (i.e. x_0 is determined only by F). Thus the asymptotic solution vanishes everywhere on a cylinder whose radius and half-length exceed those of the actual cylinder by x_0. It is not clear how well this solution treats the neutron distribution near the edges. It might equally well have been assumed that the surface on which the asymptotic solution vanishes is that surface, all points of which are at a distance x_0 *from the nearest point of the actual cylinder of material.* This surface is a cylinder with its edges rounded off to the shape of a toroid. The solution of the wave equation with this boundary condition is much more complicated than that first tried. Since this uncertainty in the treatment of the corners exists it seems unprofitable to include in the recipe the further complication of taking into account the effect of the transverse variation of the solution on the end-point. Experience with the cube (cf. Section 3) indicates that the error made in neglecting the transverse wave is of the same order of magnitude as the error in the treatment of the corners and edges. Since both the transverse-wave effect and the effect of the inaccuracy in the treatment of edges and corners are small, it is to be expected that this recipe for the treatment of finite cylinders will be fairly accurate. A number of special cases were treated also by the variation method and no discrepancy greater than a few tenths of one percent was found. The details of both treatments and a comparison of the results

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is given in LA-31. The results of the extrapolated end-point method treatment are given here in Fig. XX.

Tamped Cylinders

For the end-point treatment of tamped finite cylinders a recipe corresponding to that of the tamped infinite cylinder can be stated. The corresponding recipe would be as follows: In each medium the asymptotic solution is a solution of the wave equation in which the scale factor, k_0, is determined by the characteristic equation. At each open boundary the condition is the vanishing of the asymptotic solution a distance x_0, defined as before, beyond the boundary. At each interface between two materials the boundary condition is the equality of the logarithmic derivatives at a distance Δx_0 into the medium of lower F. If the tamper is a concentric cylinder of the same length as the core, this recipe can be applied with reasonable ease. If the tamper extends on all sides of the cylindrical core the application of this recipe becomes very difficult since no simple solutions of the wave equations give equal solutions and derivatives at the extended boundaries. For such problems, some sort of numerical solution may prove more useful.

§ 3. RECTANGULAR SOLIDS

For untamped rectangular solids the same recipe as that used for the finite cylinder has been employed. The asymptotic solution is required to vanish at a distance x_0, again a function of F alone, beyond

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each boundary, or rather on the plane faced surface so determined. The asymptotic solution has then the form

$$\cos k_1 x \cos k_2 y \cos k_3 z, \quad k_1^2 + k_2^2 + k_3^2 = k_o^2$$

The most convenient procedure for finding solutions is to fix F and two of the linear dimensions. F determines k_o and x_o. x_o and the two linear dimensions fix k_1 and k_2, hence k_3. k_3 and x_o then determine the remaining linear dimension. Variation calculations have been performed by Olum and Davis (LA-47) for several cubes and several rectangular solids with one square cross section. The end-point and variation results for cubes are given in Fig. XVI. The discrepancies between the end-point and variation results are of the order of 1/2 to 1 percent for both cubes and rectangular solids. In an attempt to determine how this error is distributed between the various roughnesses in the treatment a few of the cubes were recalculated taking into account the effect of the transverse wave. This overcorrects the error by about a factor of two; thus the error with this correction is about as great as before. It would therefore seem that the error arising from the neglect of the transverse wave is of the same order as that from the roughness in the treatment of corners and edges. It is therefore an inconsistency to correct one of these errors without correcting both.

For a tamped rectangular solid the end-point recipe is the same as for the finite cylinder, the asymptotic Bessel function solutions having been replaced by the appropriate cosines (or hyperbolic functions in regions where F is less than one). Here too the actual application of the boundary condition between the core and tamper may be very tedious.

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§4. COMPLICATED SHAPES

The exceedingly accurate results of the end-point method in treating slabs, spheres, cylinders (infinite and finite), and rectangular solids suggests that it can be extended to more complicated shapes. In particular, its success in the cases of cylinders and rectangular solids, where we have not found a rigorous theoretical motivation for it, gives considerable support to our assertion that the extrapolated end-point distance can be determined only from F and is independent of the particular symmetries of the boundaries; i.e. it is the same function of F for all shapes -- slabs, shperes, cylinders, rectangular solids, ice cream cones, etc.

Hence we enunciate the following recipe for the extrapolated end-point method, which can be applied to any shaped solid in which all surfaces are exterior surfaces; i.e. no part of a surface can see another part: hollow objects and objects having sawtoothed surfaces are excluded: In each medium (definite F value) the asymptotic solution, which is a solution of the wave equation with the magnitude of its propagation vector, k_o, determined by the characteristic equation, is established. (It is assumed that the thickness of the medium is not small compared to a mean free path). At all open boundaries this asymptotic wave-equation solution is taken to vanish at an extrapolated end-point distance x_o (a function of F alone) beyond the boundary. At each interface between two mate-ials, the boundary condition is the equality of the logarithmic derivatives of the two solutions at a distance Δx_o into the medium of lower F. The values of x_o and Δx_o are the values belonging to the plane problem of the same F. (Cf. Figs. VI, VII, VIII for the Milne kernel, Fig. XV for the Gauss Kernel.)

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One shape to which this recipe is easily applicable is an untamped chunk of material bounded by two surfaces, one plane and one spherical, so situated that the expanded surface on which the asymptotic solution is to vanish is a hemisphere. The wave-equation solution with this boundary condition is $\frac{J_{3/2}(k_0 r)}{\sqrt{r}} P_1(\cos\theta)$. If, for example, we take $F = 1.4$ then k_0 is 1.261 and x_0 is .5084. The first root of $J_{3/2}$ occurs at an argument of 4.4936, hence at a radius of 3.564. Diminishing this by x_0 leaves 3.056. The volume of the resulting "half-loaf" is then 45.0 which is 38 percent greater than the volume of the critical untamped sphere of the same F-value. If we compare this result with the minimum volume for a finite cylinder (at a length slightly less than the diameter) or the volume for a cube, both about 5 or 6 percent greater than the volume for a sphere, it is seen that the excess volume increases first slowly, then more rapidly, as the departure from spherical shape increases.

A more general shape of which the above is a special case to which this method can be applied with reasonable ease is that of the untamped "ice cream cone", i.e. a convex mass bounded by a cone and capped by part of a sphere. The radius of curvature of the spherical cap and the length of the cone may not be chosen independently but are related through the angle of the cone and the value of F. The angle of the cone may be anything between 0 and $\pi/2$ in co-latitude. In general the order of the Legendre and Bessel functions will not be simple (e.g. integral or half-integral).

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In general, any shape can be treated by this method with reasonable ease if the surface obtained by expanding in this way by one extrapolated end-point (a function of F) is a surface on which a known wave function first vanishes.

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CHAPTER VI. THE EVALUATION OF EQUIVALENT CONSTANTS

In Chapter II, Section 2, we set up the integral equation for the multiplication and diffusion of neutrons. It was there assumed that all processes are isotropic and that the neutrons are monochromatic. In problems of physical interest these assumptions are not justified. As the exact integral equations, taking into account both the anistropy of scattering and the spread in energies, is much more difficult to solve, we look for appropriate average constants to introduce into the simpler integral equation which will take account of these effects. Since the size is determined primarily by k_o, the root of the characteristic equation, and only secondarily by the extrapolated end-point, we choose the equivalent constants to give k_o correctly and disregard any effect on x_o other than that of k_o.

The Velocity- and Angle-Dependent Integral Equation

The full integral equation taking both effects into account is

$$n(\underline{r},\underline{v},t) = \int d\underline{v}' \, ds \, n(\underline{r} - s\hat{v},\underline{v}',t - \frac{s}{|v|}) \, e^{-s\sigma(|v|)}$$

$$\left[\sigma_{sc}(\underline{v}' \rightarrow \underline{v}) + \nu \, \sigma_f(|\underline{v}'|) \, \chi(\underline{v})\right] \qquad (6.1)$$

where $n(\underline{r},\underline{v},t)$ is the density of neutrons at point $\underline{r}$, velocity $\underline{v}$, and

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at time t. Here $\hat{v}$ denotes a unit vector in the direction of the vector $\underline{v}$, s is the distance from the point $\underline{r}$ in the direction $\underline{v}$, $\sigma(|v|)$ is the total probability of scattering or fission per unit length at velocity v. $\sigma_{sc}(\underline{v}'\rightarrow\underline{v})$ is the probability per unit length of the scattering of a neutron of velocity $\underline{v}'$ into a unit velocity volume element at $\underline{v}$. $\sigma_f(|v|)$ is the fission probability per unit length and $\chi(\underline{v})$ the fission spectrum.

Equivalence of the Boltzman Equation and the Integral Equation

This full form of the integral equation may be derived from the Boltzman equation as follows:

$$\frac{\partial n(\underline{r},\underline{v},t)}{\partial t} + (\underline{v}\cdot\underline{\nabla})\, n(\underline{r},\underline{v},t)$$

$$= \int d\underline{v}'\; |v'|\; \sigma(\underline{v}'\rightarrow\underline{v})\; n(\underline{r},\underline{v},t) - |v|\; n(\underline{r},\underline{v},t)\; \sigma(|v|) \qquad (6.2)$$

where

$$\sigma(\underline{v}'\rightarrow\underline{v}) = \sigma_{sc}(\underline{v}'\rightarrow\underline{v}) + \nu\, \sigma_f(|v|)\, \chi(\underline{v})$$

define s as before so that $s = 0$ at r, increasing in the direction of $-\underline{v}$.

$$\underline{v}\cdot\underline{\nabla} = -|v|\frac{\partial}{\partial s}$$

define $\xi = \dfrac{-s+vt}{2}$, $\eta = \dfrac{-s-vt}{2}$,

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then

$$-s = \xi + \eta, \qquad vt = \xi - \eta$$

$$\frac{\partial}{\partial \xi} = \frac{1}{v}\frac{\partial}{\partial t} - \frac{\partial}{\partial s}$$

$$\frac{\partial}{\partial \xi}\, n(\underline{r},\underline{v},t) + \sigma(|v|)\, n(\underline{r},\underline{v},t) = \int d\underline{v}'\, \frac{|v'|}{|v|}\, \sigma(\underline{v}' \to \underline{v})\, n(\underline{r},\underline{v}',t)$$

$$\frac{\partial}{\partial \xi}\left[e^{\sigma(|v|)\xi}\, n(\underline{r},\underline{v},t)\right] = e^{\sigma(|v|)\xi} \int d\underline{v}'\, \frac{v'}{v}\, \sigma(\underline{v}' \to \underline{v})\, n(\underline{r},\underline{v}',t)$$

We integrate in ξ from $s = s_{max}$, $t' = t - \frac{s_m}{v}$, i.e. from $\xi = \frac{vt - 2s_m}{2}$ to $\frac{vt}{2}$

$$e^{\sigma(v)\xi}\, n(\underline{r},\underline{v},t) = \int_{(-2s_m + vt)/2}^{vt/2} d\xi'\, e^{\sigma(v)\xi'} \int d\underline{v}'\sigma(\underline{v}' \to \underline{v})\, \frac{v'}{v}\, n(\underline{r}',\underline{v}',t)$$

since $n(\underline{r},\underline{v},t) = 0$ at s_m

$$n(\underline{r},\underline{v},t) = \int_0^{s_m} ds'\, e^{-\sigma(v)s'} \int d\underline{v}'\, \frac{v'}{v}\, \sigma(v' \to v)\, n(r - s'\hat{v},\underline{v}',t - \frac{s'}{v}) \tag{6.1}$$

The solutions of equation (6.1) in full open space have factorable space, time, and velocity dependences.

$$n(\underline{r},\underline{v},t) = e^{i\underline{k}\cdot\underline{r}}\, e^{\gamma t}\, n(\underline{v})$$

Here $n(\underline{v})$ depends only on the magnitude of $\underline{v}$ and the cosine, μ, of the

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angle it makes with the direction of $\underline{k}$. Then (6.1) becomes

$$n(v,\mu) = \int_0^\infty ds\; e^{-\sigma(v)s - is\mu k - \gamma s/v} \int dv'\, d\mu'\, \sigma(v',\mu' \to v,\mu)\, \frac{v'}{v}\, n(v',\mu')$$

$$= \frac{1}{\sigma(v) + \gamma/v + i\mu k} \int dv'\, d\mu'\, \sigma(v',\mu' \to v,\mu)\, \frac{v'}{v}\, n(v',\mu') \qquad (6.3)$$

We study separately the effect of anisotropy of scattering and of velocity spread.

§ 1. ANISOTROPIC SCATTERING

We here assume that the neutrons have only one velocity, say unity. Then (6.3) becomes

$$n(\mu) = \frac{1}{\sigma + \gamma + ik\mu} \int d\mu'\, \frac{\sigma(\mu' \to \mu)}{2}\, n(\mu')$$

Fission will still be assumed isotropic, scattering will be assumed to depend on μ, the cosine of the scattering angle, as

$$\sigma_{sc}(\mu) = \sigma_o + \sum_n \sigma_n\, P_n(\mu)$$

then

$$\sigma(\mu) = (\sigma_o + \nu\sigma_f) + \sum_n \sigma_n\, P_n(\mu)$$

$$= \sigma(1 + p) + \sum_n \sigma_n\, P_n(\mu)$$

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where $\sigma = \sigma_o + \sigma_f$ and $f = (\nu - 1)\,\sigma_f/(\sigma_o + \sigma_f)$. If the original direction makes an angle cosine, μ', with the polar axis, we expand the Legendre polynomials in terms of spherical harmonics in μ and μ'. Averaging over the azimuth gives

$$\sigma(\mu' \to \mu) = \sigma(1 + f) + \sum_{n=1}^{\infty} \sigma_n P_n(\mu') P_n(\mu)$$

Taking $\quad n(\mu) = \sum_{r=0}^{\infty} n_r P_r(\mu)$

$$= \frac{1/2}{\sigma + \gamma + ik\mu} \int_{-1}^{1} d\mu' \left[\sigma(1 + f) + \sum_{n=1}^{\infty} \sigma_n P_n(\mu') P_n(\mu)\right] \sum_{r=0}^{\infty} n_r P_r(\mu')$$

$$= \frac{1}{\sigma + \gamma + ik\mu} \left[\sigma(1 + f) n_o + \sum_{r=1}^{\infty} \frac{\sigma_r P_r(\mu) n_r}{2r + 1}\right]$$

Multiplying by $P_s(\mu)$ and integrating over μ gives

$$\frac{2}{2s + 1}\, n_s = \sum_{r=0}^{\infty} \frac{n_r \sigma_r (1 + f\delta_{ro})}{2r + 1} \int_{-1}^{1} \frac{d\mu\, P_r(\mu) P_s(\mu)}{\sigma + \gamma + ik\mu}$$

$$= 2 \sum_{r=0}^{s} \frac{n_r \sigma_r (1 + f\delta_{ro})}{2r + 1}\, \frac{1}{ik}\, P_r\left(\frac{\sigma + \gamma}{ik}\right) Q_s\left(\frac{\sigma + \gamma}{ik}\right)$$

$$+ 2 \sum_{r=s+1}^{\infty} \frac{n_r \sigma_r}{2r + 1}\, \frac{1}{ik}\, P_s\left(\frac{\sigma + \gamma}{ik}\right) Q_r\left(\frac{\sigma + \gamma}{ik}\right) \qquad (6.4)$$

If σ_r may be neglected for r greater than some r_o, which will be the case for any reasonable scattering distribution, then the equations (6.4)

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beyond r_0 may be discarded. The remaining equations are a finite set of linear equations which can be solved for f as a function of the remaining variables. Taking $r_0 = 2$ gives

$$\frac{1 + \gamma/\sigma}{1 + f} = \frac{\dfrac{\tan^{-1}\kappa}{\kappa} - \dfrac{\phi_1}{\kappa^2}\left(1 - \dfrac{\tan^{-1}\kappa}{\kappa}\right) + \dfrac{\phi_1\phi_2}{4\kappa^4}\left(1 - \dfrac{3}{\phi_1}\right)\left[(3 + \kappa^2)\dfrac{\tan^{-1}\kappa}{\kappa} - 3\right]}{1 - \dfrac{\phi_1}{\kappa^2}\left(1 - \dfrac{\tan^{-1}\kappa}{\kappa}\right) - \dfrac{\phi_2}{4\kappa^2}(3 + \kappa^2)\left[(3 + \kappa^2)\dfrac{\tan^{-1}\kappa}{\kappa} - 3\right] + \dfrac{\phi_1\phi_2}{4\kappa^4}\left[(3 + \kappa^2)\dfrac{\tan^{-1}\kappa}{\kappa} - 3\right]} \tag{6.5}$$

where

$$\kappa = k/(\sigma + \gamma), \quad \phi_n = \sigma_n/(\sigma + \gamma)$$

This expression for k has been evaluated for several values of γ for $f = .5$ with an assumed dependence of cross section on angle which is plausible for the scattering of neutrons around 2 Mev by heavy nuclei (see Fig. XXI). The values of k resulting were compared with those obtained from the simple characteristic equation

$$\frac{1 + \gamma/\sigma}{1 + f} = \tan^{-1}(k/\sigma + \gamma)/(k/\sigma + \gamma),$$

in which σ has been taken throughout to be the transport average of the cross section, $\sigma = (1/3)\sigma_1$. The transport average was also used in the definition of f, $(\nu - 1)\sigma_f/\sigma$. The two values of k agree to about one percent throughout the significant range of γ. Since our knowledge of the various cross sections and their angular dependence is fairly rough, this would indicate that sufficiently accurate results may be obtained by using only the transport cross section throughout the problem.

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§2. VELOCITY DEPENDENCE

To study the effects of velocity spread we take all cross sections in (6.3) isotropic, or what is approximately equivalent, the transport averages are used throughout.

$$n(v,\mu) = \frac{1}{\sigma(v) + \gamma/v + ik\mu} \; \frac{1}{2} \int dv' \, \frac{v'}{v} \, \sigma(v' \to v) \int d\mu' \, n(v',\mu')$$

$$n(v) = \int_{-1}^{1} d\mu \, n(v,\mu) = \frac{1}{2} \int_{-1}^{1} \frac{d\mu}{\sigma(v) + \gamma/v + ik\mu} \int dv' \, \frac{v'}{v} \, \sigma(v' \to v) \, n(v')$$

$$g(v) = \frac{1}{k} \tan^{-1} \frac{k}{\sigma(v) + \gamma/v} \int dv' \, \sigma(v' \to v) \, g(v') \qquad (6.6)$$

where $g(v) = vn(v)$

This equation (6.6) can be solved by iteration accompanied by readjustment at each stage of the constants entering. With certain choices of the form of the cross section, $\sigma(v' \to v)$, it can be solved analytically. One such is the following, owing to R. Feynman:

$\sigma(v' \to v)$ will consist of three parts, elastic and inelastic scattering cross sections and ν times the fission cross section. The elastic cross section is $\sigma_e(v') \, \delta(v' - v)$. The fission cross section is factorable, $(1/\nu) \, \sigma_f(v') \chi(v)$ (where χ is normalized to make $\int_0^\infty dv \, \chi(v) = \nu$) The inelastic scattering cross section will be taken in the form $\sigma_i(v') \beta(v)$ for $v \leq v'$, zero elsewhere. In this form the normalization

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constant has been absorbed in $\sigma_i(v')$. Thus the total cross section is

$$\sigma(v) = \sigma_e(v) + \sigma_f(v) + \sigma_i(v) \int_0^v dv' \beta(v')$$

Here the only restrictive assumption is that the velocity spectrum of inelastically scattered neutrons depends on the initial velocity only through a scale factor and the position of the high energy cut-off. With this assumption (6.6) becomes

$$\frac{k}{\tan^{-1} k/(\sigma(v) + \gamma/v)}\, g(v) = \sigma_e(v)\, g(v) + \chi(v) \int_0^\infty dv'\, g(v')\, \sigma_f(v') + \beta(v) \int_v^\infty dv'\, g(v')\, \sigma_i(v')$$

Defining

$$F(v) = \frac{g(v)}{\int_0^\infty dv'\, g(v')\, \sigma_f(v')}, \qquad \beta(v)\, G(v) = \frac{k}{\tan^{-1} k/(\sigma(v) + \gamma/v)} - \sigma_e(v)$$

$$\left[\frac{k}{\tan^{-1} k/(\sigma(v) + \gamma/v)} - \sigma_e(v)\right] \frac{F(v)}{\beta(v)} = \frac{\chi(v)}{\beta(v)} + \int_v^\infty dv'\, \sigma_i(v')\, F(v')$$

$$= G(v)\, F(v)$$

differentiating with respect to v

$$G'(v)\, F(v) + G(v)\, F'(v) = \left(\frac{\chi(v)}{\beta(v)}\right)' - \sigma_i(v)\, F(v)$$

The solution of this first order linear differential equation is

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$$F(v) = e^{-\int_c^{v}(dv'/G)(G' + \sigma_i)} \int_c \frac{dv'}{G}\left(\frac{\chi}{\beta}\right)' e^{\int_c^{v'}(dv''/G)(G' + \sigma_i)} \tag{6.7}$$

The definition of $F(v)$ imposes on it the condition

$$\int_c^{\infty} dv\, F(v)\, \sigma_f(v) = 1 \tag{6.8}$$

Any one of the constants entering into the determination of $F(v)$, e.g. k or γ, may be chosen so as to satisfy this condition.

A very much cruder model than this, but one giving more insight into the effect of the velocity spread, is the following. We assume that the total cross section, $\sigma(\underline{v}'\rightarrow\underline{v})$, as it occurs in (6.6) is factorable.

$$\sigma(\underline{v}'\rightarrow\underline{v}) = \sigma(\underline{v}')\Lambda(\underline{v})(1 + f)$$

where $\Lambda(\underline{v})$ is normalized to unity. Then

$$g(v) = \frac{1}{k}\tan^{-1}\frac{k}{\sigma(v) + \gamma/v}\,(1 + f)\Lambda(v)\int dv'\,\sigma(v')\,g(v')$$

$$(1 + f)\int dv\,\frac{\sigma(v)\Lambda(v)}{k}\,\tan^{-1}\frac{k}{\sigma(v) + \gamma/v} = 1$$

or

$$\int dv\,\Lambda(v)\left[\frac{1 + f}{1 + \dfrac{\gamma/v}{\sigma(v)}}\,\tan^{-1}\frac{k}{\sigma(v) + \frac{\gamma}{v}}\Bigg/\frac{k}{\sigma(v) + \frac{\gamma}{v}}\right] = 1 \tag{6.9}$$

[redacted]

The bracketed expression in the integrand will be recognized as the function which the characteristic equation equates to unity if only one velocity is represented. $\Lambda(v)$ is the velocity spectrum (not as observed at one instant but as produced in each collision). Thus for this model the characteristic equation must in this sense be satisfied in the mean. In order to see the type of effect produced by this averaging we took $\sigma(v)$ constant and $\Lambda(v)$ uniform in the three-dimensional velocity space below a definite energy and zero above. The integral occurring in (6.9) can then be evaluated analytically. This gives a relationship between f and γ. For each value of γ there will exist an average velocity, say v_γ, which makes the bracketed expression unity for these values of γ and f. For very small γ this average velocity must be the harmonic average. The result of this calculation is presented in Fig. XXII. It is seen there that for a sizable range in γ, v_γ differs only slightly from the harmonic mean, 2/3. This suggests that in problems involving not too great a spread in energies, e.g. in the metal gadget, the time scale is determined primarily by the harmonic mean velocity of the neutrons emerging from the various [redacted] of collisions. The above argument is, of course, exceedingly rough and [redacted] great reliance should [redacted] placed on its result. A good solution to the [redacted] awaits the develop[redacted] of a satisfactory many-velocity theory.

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